1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \cos x \cos y$. Find the second-order Taylor polynomial of $f\ldots$

(a) $\ldots$at $(0,0)$;

$$f(0,0) = 1 \times 1 = 1$$

$$\nabla f(0,0) = (-\sin x \cos y, -\cos x \sin y)\Big|_{(0,0)} = (0,0)$$

$$Hf(0,0) = \begin{pmatrix} -\cos x \cos y & \sin x \sin y \\ \sin x \sin y & -\cos x \cos y \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

So the 2^{nd}-order Taylor polynomial of f at $(0,0)$ is

$$Q(x,y) = 1 + (0,0)\cdot(x,y) + \frac{1}{2}(x \; y)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$$

[Note $z = Q(x,y)$ is an elliptic paraboloid!]

(b) $\ldots$at $(\frac{\pi}{2}, \frac{\pi}{2})$;

$$f\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) = 0 \qquad Hf\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\nabla f\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) = (0,0)$$

$$\Rightarrow Q(x,y) = 0 + (0,0)\cdot\left(x-\tfrac{\pi}{2}, y-\tfrac{\pi}{2}\right)$$

$$+ \frac{1}{2}\left(x-\tfrac{\pi}{2} \;\; y-\tfrac{\pi}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x-\pi/2 \\ y-\pi/2 \end{pmatrix}$$

$$= \frac{1}{2}\cdot 2\left(x-\tfrac{\pi}{2}\right)\left(y-\tfrac{\pi}{2}\right)$$

$$= \left(x-\tfrac{\pi}{2}\right)\left(y-\tfrac{\pi}{2}\right).$$

[Note $z = Q(x,y)$ is a hyper-bolic paraboloid centred at $(\pi/2, \pi/2, 0)$.]

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x,y) = 3 - 2x + 4y - z$.

(a) Find the first-order Taylor polynomial of f at $(0,0,0)$.

$f(0,0,0) = 3, \qquad \nabla f(0,0,0) = (-2, 4, -1)$

$$\Rightarrow L(x,y,z) = 3 + (-2)(x-0) + 4(y-0) + (-1)(z-0)$$

$$= 3 - 2x + 4y - z$$

$(= f(x,y,z)$ ... ooh...)

(b) Find the second-order Taylor polynomial of f at $(0,0,0)$.

$$Hf = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ since } f_x, f_y, f_z \text{ are constant}$$

$$\Rightarrow Q(x,y,z) = L(x,y,z) + \underbrace{\vec{x}^T Hf(0,0,0)\,\vec{x}}_{=0}$$

$$= 3 - 2x + 4y - z$$

$(= f(x,y,z)$ ... ooh ...)

(c) What's going on?

f is a linear polynomial!

So the first-order Taylor polynomial is already as good an approximation to f as we could hope for.

($\Rightarrow$ All higher-order Taylor polynomials are zero.)

3. For each of the following statements, determine whether it is true or false.

(a) If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$, and the second-order Taylor polynomial of f is the constant zero function, then f is the constant zero function.

False! Let $f(x,y) = x^3$. Then

$f(0,0) = 0$, $Df(0,0) = (3x^2, 0)\big|_{(0,0)} = (0,0)$

and $Hf(0,0) = \begin{pmatrix} 6x & 0 \\ 0 & 0 \end{pmatrix}\Big|_{(0,0)} = (0,0)$

$\Rightarrow Q(x,y) = 0$ for all x, y.

(b) If $Q(x,y,z)$ is the second-order Taylor polynomial of a function $f : \mathbb{R}^3 \to \mathbb{R}$ at a point where f is differentiable, then $\dfrac{\partial^3 Q}{\partial^2 x \partial z} = 0$.

True! Q is a polynomial of degree 2 so when differentiated 3 times we obtain a value of 0.

(c) If $L(x,y)$ and $Q(x,y)$ are the first- and second-order Taylor polynomials of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at a point (a,b), then the graph $z = L(x,y)$ is the tangent plane to the graph $z = Q(x,y)$ at $(x,y) = (a,b)$.

True! The tangent plane to $z = Q(x,y)$ at $(a, b, \underbrace{Q(a,b)}_{= f(a,b)})$ is

$$z = Q(a,b) + \nabla Q(a,b) \cdot (x-a, y-b)$$
$$= f(a,b) + Df(a,b) \cdot (x-a, y-b)$$

tangent plane to f

Since $Q_x = f_x$ and $Q_y = f_y$ at (a,b)