Math 290-2 Class 22

Friday 1st March 2019

Taylor polynomials and approximation

Consider a function $f:\mathbb{R}^2\to\mathbb{R}$ that is differentiable at a point $\mathbf{a}=(a,b)$. The tangent plane to f at $\mathbf{a}$ is itself the graph of a function $L:\mathbb{R}^2\to\mathbb{R}$, namely the function

$$L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b) \quad (=f(\mathbf{a})+\nabla f(\mathbf{a})\cdot(\mathbf{x}-\mathbf{a}))$$

This function serves as a *linear approximation* to f at (a,b):

- L is 'linear' since it is a linear polynomial in x and y;
- L is an 'approximation' to f at (a,b) since $L(x,y)\approx f(x,y)$ when $(x,y)\approx(a,b)$.

Observe that $\nabla L(a,b)=(f_x(a,b),f_y(a,b))=\nabla f(a,b)$.

The n^{th}**-order Taylor polynomial** of f at (a,b) is the degree n polynomial $T:\mathbb{R}^2\to\mathbb{R}$ such that T and all of its partial derivatives up to the n^{th} order agree with those of f at a. The higher the order of the Taylor polynomial, the better the approximation to f.

In particular:

- The first-order Taylor polynomial of f at (a,b) is the linear polynomial we discussed above:

$$L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

- The second-order Taylor polynomial of f at (a,b) is the quadratic polynomial

$$Q(x,y)=L(x,y)+\frac{1}{2}\left(f_{xx}(x-a)^2+2f_{xy}(a,b)(x-a)(y-b)+f_{yy}(a,b)(y-b)^2\right)$$

  Notice that $Q(a,b)=f(a,b)$ and the first- and second-order partial derivatives of Q are equal to those of f at the point (a,b). We say Q is a *quadratic approximation* to f at (a,b).

Recall that the **Hessian matrix** of $f:\mathbb{R}^2\to\mathbb{R}$ at (a,b) is given by

$$Hf(a,b)=\begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b)\end{pmatrix}$$

We can more succinctly express the second-order Taylor polynomial of f at $\mathbf{a}$ as follows:

$$Q(\mathbf{x})=\underbrace{f(\mathbf{a})\ +\ \nabla f(\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})}_{=L(\mathbf{x})}+\frac{1}{2}(\mathbf{x}-\mathbf{a})^T Hf(\mathbf{a})(\mathbf{x}-\mathbf{a})$$

This formula generalises directly to functions $\mathbb{R}^n\to\mathbb{R}$.

1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \cos x \cos y$. Find the second-order Taylor polynomial of $f \ldots$

(a) $\ldots$ at $(0,0)$;

(b) $\ldots$ at $(\frac{\pi}{2}, \frac{\pi}{2})$;

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x,y) = 3 - 2x + 4y - z$.

(a) Find the first-order Taylor polynomial of f at $(0,0,0)$.

(b) Find the second-order Taylor polynomial of f at $(0,0,0)$.

(c) What's going on?

3. For each of the following statements, determine whether it is true or false.

(a) If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$, and the second-order Taylor polynomial of f is the constant zero function, then f is the constant zero function.

(b) If $Q(x,y,z)$ is the second-order Taylor polynomial of a function $f : \mathbb{R}^3 \to \mathbb{R}$ at a point where f is differentiable, then $\dfrac{\partial^3 Q}{\partial^2 x \partial z} = 0$.

(c) If $L(x,y)$ and $Q(x,y)$ are the first- and second-order Taylor polynomials of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at a point (a,b), then the graph $z = L(x,y)$ is the tangent plane to the graph $z = Q(x,y)$ at $(x,y) = (a,b)$.