Math 290-2 Class 21

Monday 25th February 2019

Directional derivatives

The **directional derivative** $D_{\mathbf{u}} f(\mathbf{a})$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $\mathbf{a}$ in a given direction (unit vector) $\mathbf{u}$ is defined by

$$D_{\mathbf{u}}(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h}$$

Notice that for a function $f : \mathbb{R}^2 \to \mathbb{R}$ we have

$f_x(a,b) = D_{\mathbf{i}} f(a,b)$ and $f_y(a,b) = D_{\mathbf{j}} f(a,b)$

Fun fact: if f is differentiable at $\mathbf{a}$, then $\boxed{D_{\mathbf{u}} f(\mathbf{a}) = \mathbf{u} \cdot \nabla f(\mathbf{a})}$ (so '$D_{\mathbf{u}} = \mathbf{u} \cdot \nabla$'). In particular:

$$\boxed{D_{\mathbf{u}} f(\mathbf{a}) = \|\nabla f(\mathbf{a})\| \cos\theta}$$

where θ is the angle between $\nabla f(\mathbf{a})$ and $\mathbf{u}$ (with $0 \leqslant \theta \leqslant \pi$).

Some fun consequences:

(i) $D_{\mathbf{u}} f(\mathbf{a})$ is maximised when $\mathbf{u}$ points in the same direction as $\nabla f(\mathbf{a})$—thus $\nabla f(\mathbf{a})$ points in the direction of fastest increase of f;

(ii) $D_{\mathbf{u}} f(\mathbf{a})$ is minimised when $\mathbf{u}$ points in the opposite direction from $\nabla f(\mathbf{a})$—thus $-\nabla f(\mathbf{a})$ points in the direction of fastest decrease of f;

(iii) $D_{\mathbf{u}} f(\mathbf{a}) = 0$ when $\mathbf{u} \perp \nabla f(\mathbf{a})$.

In fact, (iii) implies that:

- For a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point (a,b), the vector $\nabla f(a,b)$ is perpendicular to (the tangent line to) the level curve of f at (a,b);

- For a function $f : \mathbb{R}^3 \to \mathbb{R}$ and a point (a,b,c), the vector $\nabla f(a,b,c)$ is perpendicular to (the tangent plane to) the level surface of f at (a,b,c).

1. Find the direction(s) in which the function $f(x,y) = e^{x+y}(x^2+y^2)$ is increasing most rapidly at the point $(1,2)$.

2. Find an equation for the tangent plane to the surface $2x^2+y^2-z^2=4$ at the point $(2,0,2)$.

3. Define functions $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$\mathbf{g}(x,y) = \left(\frac{e^{x+y} + e^{x-y}}{2}\,,\, \frac{e^{x+y} - e^{x-y}}{2} \right) \quad \text{and} \quad h(s,t) = 2st$$

Find $D_{(-1,2)} f(2,1)$, where $f(x,y) = h(\mathbf{g}(x,y))$.

4. Peeve the guinea pig is standing on a steep Andes mountain. Conveniently, the mountain looks just like the elliptic paraboloid $3x^2 + 7y^2 + 5z = 20$. Owing to her short legs, the steepest grade that Peeve can climb up the mountain is $\frac{1}{5}$.

Given that Peeve's (x, y)-coordinates are $(1, 1)$ and she is facing in the direction of steepest ascent, find the smallest angle that Peeve must turn in order to be able to ascend the mountain.