Math 290-2 Class 20

Monday 25th February 2019

Directional derivatives

The **directional derivative** $D_{\mathbf{u}} f(\mathbf{a})$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $\mathbf{a}$ in a given direction (unit vector) $\mathbf{u}$ is defined by

$$D_{\mathbf{u}}(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h}$$

Notice that for a function $f : \mathbb{R}^2 \to \mathbb{R}$ we have

$$f_x(a,b) = D_{\mathbf{i}} f(a,b) \text{ and } f_y(a,b) = D_{\mathbf{j}} f(a,b)$$

Fun fact: if f is differentiable at $\mathbf{a}$, then $\boxed{D_{\mathbf{u}} f(\mathbf{a}) = \mathbf{u} \cdot \nabla f(\mathbf{a})}$ (so '$D_{\mathbf{u}} = \mathbf{u} \cdot \nabla$'). In particular:

$$\boxed{D_{\mathbf{u}} f(\mathbf{a}) = \|\nabla f(\mathbf{a})\| \cos \theta}$$

where θ is the angle between $\nabla f(\mathbf{a})$ and $\mathbf{u}$ (with $0 \leqslant \theta \leqslant \pi$).

Some fun consequences:

(i) $D_{\mathbf{u}} f(\mathbf{a})$ is maximised when $\mathbf{u}$ points in the same direction as $\nabla f(\mathbf{a})$—thus $\nabla f(\mathbf{a})$ points in the direction of fastest increase of f;

(ii) $D_{\mathbf{u}} f(\mathbf{a})$ is minimised when $\mathbf{u}$ points in the opposite direction from $\nabla f(\mathbf{a})$—thus $-\nabla f(\mathbf{a})$ points in the direction of fastest decrease of f;

(iii) $D_{\mathbf{u}} f(\mathbf{a}) = 0$ when $\mathbf{u} \perp \nabla f(\mathbf{a})$.

In fact, (iii) implies that:

- For a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point (a,b), the vector $\nabla f(a,b)$ is perpendicular to (the tangent line to) the level curve of f at (a,b);
- For a function $f : \mathbb{R}^3 \to \mathbb{R}$ and a point (a,b,c), the vector $\nabla f(a,b,c)$ is perpendicular to (the tangent plane to) the level surface of f at (a,b,c).

1. Compute the directional derivative of the function $f(x,y) = x^2y + y^2x$ at $(1,2)$ in the direction of the vector $(-2,1)$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function, let $\mathbf{u} = (u,v)$ be a unit vector in $\mathbb{R}^2$, and let $\mathbf{a} = (a,b)$ in $\mathbb{R}^2$. Assuming f is differentiable at $\mathbf{a}$, use the chain rule to show that $D_{\mathbf{u}}f(\mathbf{a}) = \mathbf{u} \cdot \nabla f(\mathbf{a})$.

3. Find the direction(s) in which the function $f(x,y) = e^{x+y}(x^2 + y^2)$ is increasing most rapidly at the point $(1,2)$.

4. Find an equation for the tangent plane to the surface $2x^2 + y^2 - z^2 = 4$ at the point $(2,0,2)$.