Math 290-2 Class 19

Friday 22nd February 2019

Recall that the **Jacobian matrix** of a vector-valued function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$ matrix $D\mathbf{f}$ such that $(D\mathbf{f})_{ij} = \dfrac{\partial f_i}{\partial x_j}$ for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Writing $\partial_{x_j} f_i$ for $\partial f_i / \partial x_i$, we have:

$$D\mathbf{f} = \begin{pmatrix} \cdots & \nabla f_1 & \cdots \\ \cdots & \nabla f_2 & \cdots \\ & \vdots & \\ \cdots & \nabla f_m & \cdots \end{pmatrix} = \begin{pmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 & \cdots & \partial_{x_n} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 & \cdots & \partial_{x_n} f_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} f_m & \partial_{x_2} f_m & \cdots & \partial_{x_n} f_m \end{pmatrix}$$

The product rule

Notice that if $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$ are functions, then their dot product gives a function $\mathbf{f} \cdot \mathbf{g} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$(\mathbf{f} \cdot \mathbf{g})(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) = f_1(\mathbf{x}) g_1(\mathbf{x}) + f_2(\mathbf{x}) g_2(\mathbf{x}) + \cdots + f_m(\mathbf{x}) g_m(\mathbf{x})$$

A straightforward but tedious computation reveals that

$$\nabla(\mathbf{f} \cdot \mathbf{g})(\mathbf{x}) = \mathbf{g}(\mathbf{x})^T D\mathbf{f}(\mathbf{x}) + \mathbf{f}(\mathbf{x})^T D\mathbf{g}(\mathbf{x}) \quad \text{i.e.} \quad \boxed{\frac{\partial (\mathbf{f} \cdot \mathbf{g})}{\partial x_j}(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) \cdot \frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{x})}$$

Two special cases of this are:

- When $f, g : \mathbb{R}^n \to \mathbb{R}$, we have $\boxed{\nabla(fg)(\mathbf{x}) = g(\mathbf{x}) \nabla f(\mathbf{x}) + f(\mathbf{x}) \nabla g(\mathbf{x})}$;
- When $f, g : \mathbb{R} \to \mathbb{R}^m$, we have $\boxed{\dfrac{d}{dt}(\mathbf{f}(t) \cdot \mathbf{g}(t)) = \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t)}$.

The chain rule

The chain rule has a very simple statement in terms of Jacobians: if $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^k$ are functions, then $\boxed{D(\mathbf{g} \circ \mathbf{f}) = (D\mathbf{g})(D\mathbf{f})}$.

This implies, for example, that if $f(x,y)$ is a scalar-valued function and $x = x(s,t)$ and $y = y(s,t)$, then we have

$$Df(\mathbf{x}(s,t)) = (Df(x,y))(D\mathbf{x}(s,t)) = \begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} x_s & x_t \\ y_s & y_t \end{pmatrix} = \begin{pmatrix} f_x x_s + f_y y_s & f_x x_t + f_y y_t \end{pmatrix}$$

or, put another way:

$$\boxed{\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}$$

1. Compute the gradient vector of the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$h(x,y) = \sin(x^2 + y^2)e^{xy}$$

2. Compute $\nabla(\mathbf{f} \cdot \mathbf{g})$, where $\mathbf{f}, \mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by

$$\mathbf{f}(x,y) = (x^2 + y^2, x^2 - y^2) \quad \text{and} \quad \mathbf{g}(x,y) = (2xy, -2xy)$$

3. Given that $f(x,y,z) = x^2e^y + y^2e^z + z^2e^x$, compute the rate of change of $f(\mathbf{x})$ with respect to the distance of $\mathbf{x}$ from the z-axis when $(x,y,z) = (1,1,2)$.

4. Find the Jacobian matrix of $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the variables (s,t), where

$$\mathbf{f}(x,y) = (2xy, x^2 - y^2), \quad x(s,t) = e^s \cos t \quad \text{and} \quad y(s,t) = e^s \sin t$$

5. [Colley, §2.5 Q6] A rectangular stick of bvutter is placed in a microwave oven to melt. When the butter's length is 6 in and its square cross section on one side measures $\frac{3}{2}$ in, its length is decreasing at a rate of $\frac{1}{4}$ in min^{-1}, and its cross-sectional edge is decreasing at a rate of $\frac{1}{8}$ in min^{-1}. How fast is butter melting at that instant (i.e. what is the rate of decrease of its volume in in^3 min^{-1})?

6. Suppose $f = f(x,y,z)$, $(x,y,z) = \mathbf{x}(r,s,t)$ and $(r,s,t) = \mathbf{r}(\lambda,\mu)$.

Find expressions for f_λ and f_μ.