Math 290-2 Class 18

Wednesday 20th February 2019

Higher-order partial derivatives

The partial derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}$ are themselves functions $\mathbb{R}^n \to \mathbb{R}$, so they themselves can be differentiated... and so on. The resulting functions are called the **higher-order partial derivatives** of f.

For example, a function $f : \mathbb{R}^2 \to \mathbb{R}$ has four second-order partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

This notation is clunky, so we will instead write, respectively:

$$f_{xx}, f_{xy}, f_{yx}, f_{yy} \quad \text{or} \quad \frac{\partial^2 f}{\partial^2 x}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$$

This generalises to higher-order partial derivatives—for example

$$f_{xyyxy} = \frac{\partial^5 f}{\partial x \partial^2 y \partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right)\right)\right)$$

The partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$ assemble into an $n \times n$ matrix Hf, called the **Hessian matrix** of f; the $(i,j)^{\text{th}}$ component of Hf is $f_{x_i x_j}$. For example in $\mathbb{R}^2$:

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Clairaut's theorem (or **Schwarz's theorem**, depending on who you ask) says that if $f : \mathbb{R}^n \to \mathbb{R}$ has continuous first- and second-order partial derivatives, then the order of differentiation does not matter. This means that, for such functions† *the Hessian matrix is symmetric*—wowzer!

[†Most functions we will encounter in this course do indeed have continuous first- and second-order partial derivatives.]

Coming soon: if $\nabla f(\mathbf{a}) = \mathbf{0}$, then we can classify the nature of the critical point at $\mathbf{a}$—e.g. if it is local minimum, local maximum, or something else—by considering the definiteness of $Hf(\mathbf{a})$.

The Jacobian matrix

Vector-valued functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ can also be differentiated! Instead of just a *vector* of partial derivatives, we now get an $m \times n$ *matrix* Df of partial derivatives, called the **Jacobian matrix**:

$$D\mathbf{f} = \begin{pmatrix} \cdots & \nabla f_1 & \cdots \\ & \vdots & \\ \cdots & \nabla f_m & \cdots \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & & \vdots & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Notice that when $m = 1$ we have $Df = \nabla f$ (as a row vector), and so $\nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$.

1. Compute the second-order partial derivatives of the function $f(x,y) = \sin(x^2+y^2)$.

$f_x = 2x\cos(x^2+y^2)$, $\quad f_y = 2y\cos(x^2+y^2)$

$\Rightarrow f_{xx} = 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2)$

$f_{xy} = \frac{\partial}{\partial x}(f_y) = -4xy\sin(x^2+y^2)$

$f_{yx} = \frac{\partial}{\partial y}(f_x) = -4xy\sin(x^2+y^2)$

They're equal! Wow!

$f_{yy} = 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2)$

2. Find the Hessian matrix of $f(x,y) = x^2 - xy + y^2$.

$f_x = 2x - y$, $\quad f_y = 2y - x$

$\Rightarrow f_{xx} = 2$, $f_{xy} = -1$, $f_{yx} = -1$, $f_{yy} = 2$

They're equal! Wow!

$$\Rightarrow Hf = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

3. Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by $f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$

You may assume that

$$f_x=\begin{cases}\dfrac{3x^2y-y^3}{x^2+y^2}-\dfrac{2x^2y(x^2-y^2)}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

and

$$f_y=\begin{cases}\dfrac{x^3-3xy^2}{x^2+y^2}-\dfrac{2xy^2(x^2-y^2)}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

(a) Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

Note for $y\neq0$ we have $f_x(0,y)=\dfrac{0-y^3}{0+y^2}-\dfrac{0}{(0+y^2)^2}=-y$

& for $x\neq0$ we have $f_y(x,0)=\dfrac{x^3-0}{x^2+0^2}-\dfrac{0}{(x^2+0)^2}=x$

$$\Rightarrow f_{xy}(0,0)=\lim_{x\to0}\frac{f_y(x,0)-f_y(0,0)}{x-0}=\lim_{x\to0}\frac{x-0}{x-0}=\underline{\underline{1}}$$

$$\& \quad f_{yx}(0,0)=\lim_{y\to0}\frac{f_x(0,y)-f_x(0,0)}{y-0}=\lim_{y\to0}\frac{-y-0}{y-0}=\underline{\underline{-1}}$$

(b) Explain why $f_{xy}(0,0)\neq f_{yx}(0,0)$.

f_{xy} and f_{yx} are not continuous.

[Proving this is very tedious
but not impossible using
your Math 290-2 skills!]

4. Find the Jacobian matrix of each of the following vector-valued functions, and then compute the determinant of the Jacobian matrix in each case.

(a) $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{f}(r,\theta) = (r\cos\theta, r\sin\theta)$

$$D\vec{f} = \begin{pmatrix} \frac{\partial}{\partial r}(r\cos\theta) & \frac{\partial}{\partial\theta}(r\cos\theta) \\ \frac{\partial}{\partial r}(r\sin\theta) & \frac{\partial}{\partial\theta}(r\sin\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\Rightarrow \det(D\vec{f}) = r\cos^2\theta + r\sin^2\theta = \underline{\underline{r}}$$

(b) $\mathbf{f}(\rho,\varphi,\theta) = (\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)$

$$D\vec{f} = \begin{array}{c} \begin{pmatrix} \cos\theta\sin\varphi & \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \rho\sin\theta\cos\varphi & \rho\cos\theta\sin\varphi \\ \cos\varphi & -\rho\sin\varphi & 0 \end{pmatrix} \\ \begin{array}{ccc} \uparrow \quad\quad\quad & \quad\quad\uparrow\quad\quad & \quad\quad\quad\uparrow \\ \partial/\partial\rho \quad\quad\quad & \quad\quad\partial/\partial\varphi\quad\quad & \quad\quad\quad\partial/\partial\theta \end{array} \end{array}$$

expanding down 3rd column

$$\begin{aligned} \Rightarrow \det(D\vec{f}) &= -\rho\sin\theta\sin\varphi\left(-\rho\sin\theta\left(\cancelto{1}{\sin^2\varphi + \cos^2\varphi}\right)\right) \\ &\quad - \rho\cos\theta\sin\varphi\left(-\rho\cos\theta\left(\cancelto{1}{\sin^2\varphi + \cos^2\varphi}\right)\right) \\ &= \rho^2\sin\varphi\left(\cancelto{1}{\sin^2\theta + \cos^2\theta}\right) \\ &= \underline{\underline{\rho^2\sin\varphi}} \end{aligned}$$

[In Math 290-3, when we integrate over a curve using polar coordinates, we'll make the translation '$dxdy = r\,dr\,d\theta$', and when we integrate over a surface using spherical coordinates, we'll make the translation '$dxdydz = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$'. This is where the strange multiplicative terms come from.]