Math 290-2 Class 18

Wednesday 20th February 2019

Higher-order partial derivatives

The partial derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}$ are themselves functions $\mathbb{R}^n \to \mathbb{R}$, so they themselves can be differentiated... and so on. The resulting functions are called the **higher-order partial derivatives** of f.

For example, a function $f : \mathbb{R}^2 \to \mathbb{R}$ has four second-order partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

This notation is clunky, so we will instead write, respectively:

$$f_{xx}, f_{xy}, f_{yx}, f_{yy} \quad \text{or} \quad \frac{\partial^2 f}{\partial^2 x}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$$

This generalises to higher-order partial derivatives—for example

$$f_{xyyxy} = \frac{\partial^5 f}{\partial x \partial^2 y \partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right)\right)\right)$$

The partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$ assemble into an $n \times n$ matrix Hf, called the **Hessian matrix** of f; the $(i, j)^{\text{th}}$ component of Hf is $f_{x_i x_j}$. For example in $\mathbb{R}^2$:

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Clairaut's theorem (or **Schwarz's theorem**, depending on who you ask) says that if $f : \mathbb{R}^n \to \mathbb{R}$ has continuous first- and second-order partial derivatives, then the order of differentiation does not matter. This means that, for such functions[†] *the Hessian matrix is symmetric*—wowzer!

[[†]Most functions we will encounter in this course do indeed have continuous first- and second-order partial derivatives.]

Coming soon: if $\nabla f(\mathbf{a}) = \mathbf{0}$, then we can classify the nature of the critical point at $\mathbf{a}$—e.g. if it is local minimum, local maximum, or something else—by considering the definiteness of $Hf(\mathbf{a})$.

The Jacobian matrix

Vector-valued functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ can also be differentiated! Instead of just a *vector* of partial derivatives, we now get an $m \times n$ *matrix* Df of partial derivatives, called the **Jacobian matrix**:

$$D\mathbf{f} = \begin{pmatrix} \cdots & \nabla f_1 & \cdots \\ & \vdots & \\ \cdots & \nabla f_m & \cdots \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & & \vdots & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Notice that when $m = 1$ we have $Df = \nabla f$ (as a row vector), and so $\nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$.

1. Compute the second-order partial derivatives of the function $f(x,y) = \sin(x^2 + y^2)$.

2. Find the Hessian matrix of $f(x,y) = x^2 - xy + y^2$.

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

You may assume that

$$f_x = \begin{cases} \dfrac{3x^2y - y^3}{x^2+y^2} - \dfrac{2x^2y(x^2-y^2)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

and

$$f_y = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2} - \dfrac{2xy^2(x^2-y^2)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

(b) Explain why $f_{xy}(0,0) \neq f_{yx}(0,0)$.

4. Find the Jacobian matrix of each of the following vector-valued functions, and then compute the determinant of the Jacobian matrix in each case.

(a) $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{f}(r, \theta) = (r\cos\theta, r\sin\theta)$

(b) $\mathbf{f}(\rho, \varphi, \theta) = (\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)$

[In Math 290-3, when we integrate over a curve using polar coordinates, we'll make the translation '$dx\,dy = r\,dr\,d\theta$', and when we integrate over a surface using spherical coordinates, we'll make the translation '$dx\,dy\,dz = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$'. This is where the strange multiplicative terms come from.]