1. Consider the function $f:\mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = ||x|-|y|| - |x| - |y|$.

(a) Show that $f_x(0,0)$ and $f_y(0,0)$ exist, and compute their values.

$$f_x(0,0) = \lim_{x\to 0} \frac{(||x|-|0|| - |x| - |0|) - 0}{x - 0} = \lim_{x \to 0} 0 = 0$$

$$f_y(0,0) = \lim_{y\to 0} \frac{(||0|-|y|| - |0| - |y|) - 0}{y - 0} = \lim_{y \to 0} 0 = 0$$

[note that $|-|y|| = ||y|| = |y|$]

(b) Show that f is not differentiable at $(0,0)$.

If f were differentiable at $(0,0)$, then we'd have

$$\lim_{\vec{x}\to\vec{0}} \frac{(||x|-|y|| - |x| - |y|) - [0 + (0,0)\cdot(x,y)]}{\sqrt{x^2+y^2}} = 0$$

$f(0,0)=0$ $\quad$ $\vec{\nabla} f(0,0) = (0,0)$

$\|(x,y)-(0,0)\| = \sqrt{x^2+y^2}$

However, along the x-axis $(x,y) = (t,0)$ we get

$$\lim_{t\to 0} \frac{||t|-|0|| - |t| - |0|}{\sqrt{t^2+0^2}} = \lim_{t\to 0} 0 = 0$$

and along the line $y = x$, $(x,y) = (t,t)$ we get

$$\lim_{t\to 0} \frac{||t|-|t|| - |t| - |t|}{\sqrt{t^2+t^2}} = \lim_{t\to 0} \frac{-2|t|}{\sqrt{2}\,|t|} = -\sqrt{2}$$

The limit doesn't exist, so f is not differentiable at $(0,0)$.

2. Define $f(x,y) = x^2 + y^2$.

(a) Find the equation of the tangent plane to the graph of $f(x,y)$ at a general point (a,b).

$$\nabla f(a,b) = (2a,\ 2b) \qquad f(a,b) = a^2+b^2$$

$$\rightsquigarrow \quad z = (a^2+b^2) + (2a, 2b)\cdot(x-a, y-b)$$

$$\Leftrightarrow \quad z = a^2+b^2 + 2ax - 2a^2 + 2by - 2b^2$$

$$\Leftrightarrow \quad \boxed{z = 2ax - a^2 + 2by - b^2}$$

(b) Verify that f is differentiable at (a,b).

$$\lim_{(x,y)\to(a,b)} \frac{(x^2+y^2) - (2ax - a^2 + 2by - b^2)}{\sqrt{(x-a)^2+(y-b)^2}}$$

$\because\ x^2 - 2ax + a^2 = (x-a)^2$, etc

$$= \lim_{(x,y)\to(a,b)} \frac{(x-a)^2 + (y-b)^2}{\sqrt{(x-a)^2+(y-b)^2}}$$

$$= \lim_{(x,y)\to(a,b)} \sqrt{(x-a)^2+(y-b)^2}$$

$= \underline{\underline{0}}$ as required

(c) Explain how you could have known that f was differentiable everywhere without having to directly compute any limits at all.

$\frac{\partial f}{\partial x} = 2x$ & $\frac{\partial f}{\partial y} = 2y$ — both are continuous everywhere

$\Rightarrow f$ is d'ble everywhere.

3. For each of the following statements, determine whether it is always, sometimes or never true.

(a) Let $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ for some symmetric matrix A. Then q is differentiable everywhere.

Always $\quad q$ is polynomial in each variable x_i

$\Rightarrow \frac{\partial q}{\partial x_i}$ is polynomial in each variable

$\Rightarrow$ cts everywhere

$\Rightarrow q$ is differentiable everywhere.

(b) If $\nabla f(a,b)$ is defined, then f is differentiable at (a,b).

Sometimes

True if $f(x,y) = 0$ (constant)

False if $f(x,y) = ||x|-|y|| - |x| - |y|$ & $(a,b) = (0,0)$

(c) The function $f : \mathbb{R}^2 \to \mathbb{R}$, defined in terms of a constant k as follows, is differentiable.

$$f(x,y) = \begin{cases} \frac{2x^2+3y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ k & \text{if } (x,y) = (0,0) \end{cases}$$

Never

f isn't even continuous at $(0,0)$ since

$\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist :

$$\begin{cases} \text{Along } x\text{-axis}: & \lim_{t\to 0} \frac{2t^2}{t^2} = 2 \\ \text{Along } y\text{-axis}: & \lim_{t\to 0} \frac{3t^2}{t^2} = 3 \end{cases}$$