Math 290-2 Class 17

Monday 18th February 2019

Differentiability

What it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be differentiable at a is that

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists} \qquad \text{or equivalently} \qquad \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists}$$

The value of this limit is then equal to $f'(a)$.

For a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point (a,b) in $\mathbb{R}^2$, we have

$$\frac{\partial f}{\partial x}(a,b) = \lim_{x \to a} \frac{f(x,b) - f(a,b)}{x - a} \quad \text{and} \quad \frac{\partial f}{\partial y}(a,b) = \lim_{y \to b} \frac{f(a,y) - f(a,b)}{y - b}$$

We will see that the partial derivatives of a function $f : \mathbb{R}^2 \to \mathbb{R}$ might both exist at a point (a,b), and yet there is no meaningful notion of 'tangent plane'. We therefore need a stronger definition of what it means to be differentiable.

The criterion for a function $f : \mathbb{R} \to \mathbb{R}$ to be differentiable at a can be rearranged to give

$$\lim_{x \to a} \frac{f(x) - [f(a) + f'(a)(x-a)]}{x - a} = 0$$

Notice that $y = f(a) + f'(a)(x-a)$ is the equation of the tangent line to the curve $y = f(x)$ at $x = a$, so this limit is intuitively saying that 'the tangent line approximates the graph of $f(x)$ near $x = a$'.

Recall that, for a function $f : \mathbb{R}^2 \to \mathbb{R}$, the tangent plane to the surface $z = f(x,y)$ at $\mathbf{a} = (a,b)$, if it exists, is given by $z = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$.

We're now ready to give a definition of differentiability for functions $\mathbb{R}^2 \to \mathbb{R}$: we say f is differentiable at $\mathbf{a}$ if (the partial derivatives $f_x(\mathbf{a})$ and $f_y(\mathbf{a})$ exist and)

$$\boxed{\lim_{\mathbf{x} \to \mathbf{a}} \frac{f(\mathbf{x}) - [f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})]}{\|\mathbf{x} - \mathbf{a}\|} = 0}$$

This says that 'the tangent line approximates the graph of $f(\mathbf{x})$ near $\mathbf{x} = \mathbf{a}$'. [The very same definition applies to functions $f : \mathbb{R}^n \to \mathbb{R}$, just with 'tangent plane' replaced by 'tangent space'.]

Some useful facts:

- If f has continuous partial derivatives in a neighbourhood of $\mathbf{a}$, then f is differentiable at $\mathbf{a}$.
- If f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = ||x| - |y|| - |x| - |y|$.

(a) Show that $f_x(0,0)$ and $f_y(0,0)$ exist, and compute their values.

(b) Show that f is not differentiable at $(0,0)$.

2. Define $f(x,y) = x^2 + y^2$.

(a) Find the equation of the tangent plane to the graph of $f(x,y)$ at a general point (a,b).

(b) Verify that f is differentiable at (a,b).

(c) Explain how you could have known that f was differentiable everywhere without having to directly compute any limits at all.

3. For each of the following statements, determine whether it is always, sometimes or never true.

(a) Let $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ for some symmetric matrix A. Then q is differentiable everywhere.

(b) If $\nabla f(a,b)$ is defined, then f is differentiable at (a,b).

(c) The function $f : \mathbb{R}^2 \to \mathbb{R}$, defined in terms of a constant k as follows, is differentiable.

$$f(x,y) = \begin{cases} \frac{2x^2+3y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ k & \text{if } (x,y) = (0,0) \end{cases}$$