Math 290-2 Class 16

Friday 15th February 2019

Partial derivatives

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, we can find the rate of change of $f(x_1, x_2, \ldots, x_n)$ as just one of its variables x_i varies by differentiating f with respect to x_i and holding all other variables constant.

The resulting function $\mathbb{R}^n \to \mathbb{R}$ is called the **partial derivative** of f with respect to x_i, and can be written as $\dfrac{\partial f}{\partial x_i}$ or simply f_{x_i}. For example, if f is a function $\mathbb{R}^2 \to \mathbb{R}$:

- $\dfrac{\partial f}{\partial x}(a,b)$ gives the slope of the curve $z = f(x,b)$ at $x = a$;
- $\dfrac{\partial f}{\partial y}(a,b)$ gives the slope of the curve $z = f(a,y)$ at $y = b$.

Note that the lines containing $(a,b,f(a,b))$ and parallel to the vectors $(1,0,f_x(a,b))$ and $(0,1,f_y(a,b))$, respectively, are both tangent to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$.

These lines lie on a plane, called the **tangent plane** to the graph of f at (a,b).

The normal vector to the tangent plane is given by $(1,0,f_x(a,b)) \times (0,1,f_y(a,b)) = (f_x(a,b), f_y(a,b), -1)$, and so the equation of the tangent plane is

$$f_x(a,b)(x-a) + f_y(a,b)(y-a) - (z - f(a,b)) = 0$$

or equivalently

$$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

The gradient vector

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a point $\vec{a}$ in $\mathbb{R}^n$, the **gradient vector** of f at $\vec{a}$ is the vector ∇f defined by

$$\nabla f(\vec{a}) = (f_{x_1}(\vec{a}), f_{x_2}(\vec{a}), \ldots, f_{x_n}(\vec{a}))$$

For example if $f : \mathbb{R}^2 \to \mathbb{R}$ and (a,b) is a point in $\mathbb{R}^n$, then

$$\nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b) \ , \ \frac{\partial f}{\partial y}(a,b) \right)$$

This gives us a nice expression for the tangent plane of f, namely

$$\boxed{z = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})}$$

where we have written $\vec{a} = (a,b)$ and $\vec{x} = (x,y)$.

1. Find the partial derivatives of the following functions.

(a) $f(x,y) = x^2 + y^2$

(b) $g(x,y) = \sin(x+y)\cos(x-y)$

(c) $h(x,y) = e^{xy^2+yz^2+zx^2}$

2. For each of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ and points (a,b) in $\mathbb{R}^2$, find the equation of the tangent plane to the graph of f at (a,b). Draw a sketch if you can.

(a) $f(x,y) = x^2 + y^2$; $(a,b) = (1,1)$.

(b) $f(x,y) = x\cos y - y\sin x$; $(a,b) = (0, \frac{\pi}{4})$.

3. For each of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$, sketch the level curves of its graph and indicate the direction of $\nabla f(a,b)$ at a few points (a,b) of your choosing.

(a) $f(x,y) = x^2 + y^2$

(b) $f(x,y) = x^2 - y^2$