Math 290-2 Class 15

Wednesday 13th February 2019

Limits

Consider a function f from (some subset of) $\mathbb{R}^n$ to $\mathbb{R}$. Given a vector $\mathbf{a}$ in $\mathbb{R}^n$, the **limit** of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{a}$, if it exists, is the value ℓ that the function becomes arbitrary close to whenever $\mathbf{x}$ is an arbitrarily small positive distance from $\mathbf{a}$. In this case, we write

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \ell \quad \text{or} \quad f(\mathbf{x}) \to \ell \text{ as } \mathbf{x} \to \mathbf{a}$$

Limits do not always exist.[a] Some ways that limits can fail to exist include:

- The usual '1-dimensional' reasons, such as the denominator of a fraction tending to zero while its numerator does not.
- The function might approach multiple values depending on the 'path' along which the variable $\mathbf{x}$ approaches $\mathbf{a}$. If this is the case, a limit does not exist.

For example, if you suspect a limit of $f(x,y)$ does not exist as $(x,y) \to (0,0)$ because the limit is not 'independent of path', some suggestions include:

- Set $x = 0$ and compute the limit as $y \to 0$, and set $y = 0$ and compute the limit as $x \to 0$.
- Set $y = mx$ for some real number m and compute the limit as $x \to 0$.
- Set $y = x^k$ for some power k and compute the limit as $x \to 0$.

If any of the above limits do not equal any of the others, the limit does not exist.

As a rule of thumb, if f is built out of nice, continuous functions (such as polynomials, exponentials and trig functions) using arithmetic operations, and the denominators involved do not tend to zero, then a limit exists. If not, some more care is needed.

If you're struggling to compute a limit (or show it doesn't exist), try converting to a different system of coordinates, such as polar coordinates (in $\mathbb{R}^2$), or cylindrical or spherical coordinates (in $\mathbb{R}^3$).

A related concept is *continuity*:

- The limit $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ might not actually be equal to $f(\mathbf{a})$.
- If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$, we say f is **continuous at a**.
- If f is continuous at $\mathbf{a}$ for all $\mathbf{a}$ in its domain, we say f is **continuous**.

[a] https://youtu.be/oDAKKQuBtDo?t=45

1. For each of the following, either evaluate the limit or show it does not exist.

(a) $\displaystyle\lim_{(x,y,z)\to(0,1,\pi)} \frac{e^{x+y^2}\cos(x^2+2z)+xyz}{x^2+y^2+z^2}$.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2+3y^2}{x^2+y^2}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$ [*Colley,* §*2.2 Q15*]

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

(e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

(f) $\lim_{(x,y)\to(0,0)} \dfrac{x^2+xy+y^2}{x^2+y^2}$ [*Colley,* §*2.2 Q30*]

(g) $\lim_{(x,y)\to(0,0)} \dfrac{x^2+xy+y^2}{\sqrt{x^2+y^2}}$