Math 290-2 Class 14

Monday 11th February 2019

Quadric surfaces

A *quadric surface* is a surface in $\mathbb{R}^3$ of the form $\mathbf{x}^T A\mathbf{x} + \mathbf{b} \cdot \mathbf{x} + c = 0$, where A is a 3×3 symmetric matrix, $\mathbf{b}$ is a vector in $\mathbb{R}^3$ and c is a scalar. That is:

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + b_1x + b_2y + b_3z + c = 0$$

Some special cases include the following. [See supplemental handout for sketches.]

- Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Hyperboloid of two sheets: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Elliptic cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ — [*think about what a 'hyperbolic cone' would be*]
- Elliptic paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
- Hyperbolic paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Quadratic forms revisited (not in Colley)

Every quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ defines a quadric surface. If $q(x,y,z) = \mathbf{x}^T A\mathbf{x}$, with A symmetric, then we obtain

$$q(x,y,z) = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2$$

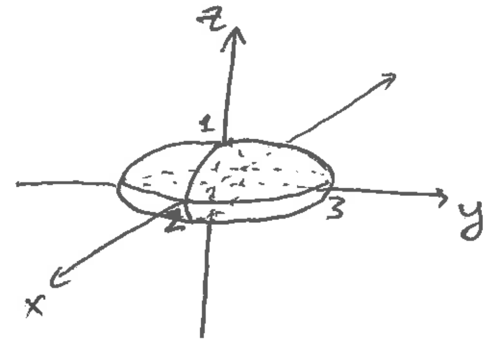
where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A and (c_1, c_2, c_3) are the coordinates of (x,y,z) with respect to the orthonormal eigenbasis $\vec{u}_1, \vec{u}_2, \vec{u}_3$.

We can then sketch $q(x,y,z) = 1$ on (c_1, c_2, c_3)-axes, just as we did for ellipses and hyperbolae in two dimensions. Note that a surface of the form $\mathbf{x}^T A\mathbf{x} = 1$ will either be an ellipsoid, a hyperboloid of one or two sheets, or an (elliptic or hyperbolic) *cylinder* (this occurs when one of the eigenvalues is zero).

1. Describe (and try to sketch) the following surfaces:

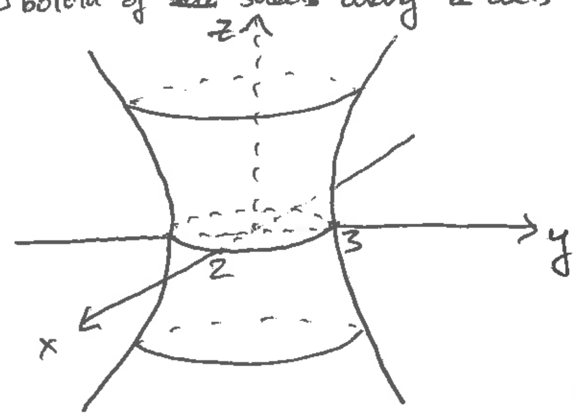
(a) $\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$

Ellipse w/ x-intercepts ± 2, y-intercepts ± 3 & z-intercepts ± 1.



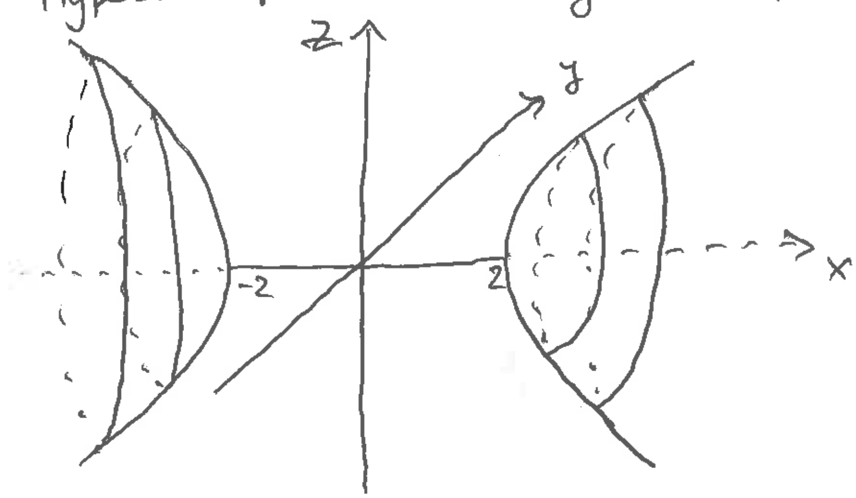
(b) $\frac{x^2}{4}+\frac{y^2}{9}-z^2=1$

Hyperboloid of ~~two~~ one sheet along z-axis w/ x-intercepts ± 2 & y-intercepts ± 3



(c) $\frac{x^2}{4}-\frac{y^2}{9}-z^2=1$

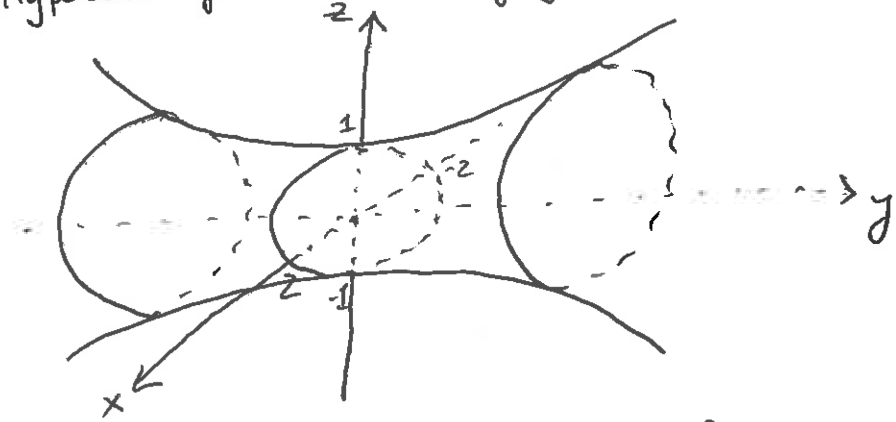
Hyperboloid of two sheets along x-axis w/ x-intercepts ± 2



Note the change of perspective in the sketch:

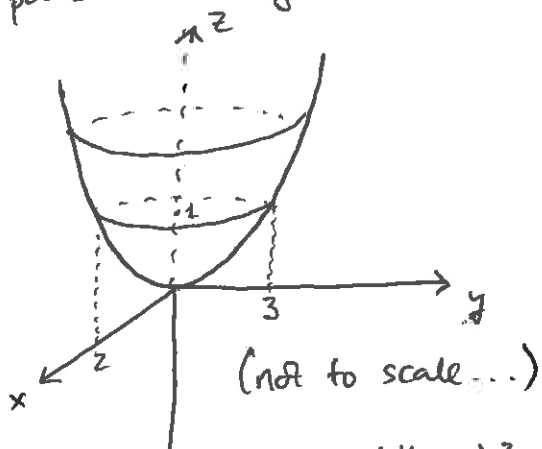
(d) $\frac{x^2}{4}-\frac{y^2}{9}+z^2=1$

Hyperboloid of one sheet along y-axis w/ x-intercepts ± 2 & z-intercepts ± 1



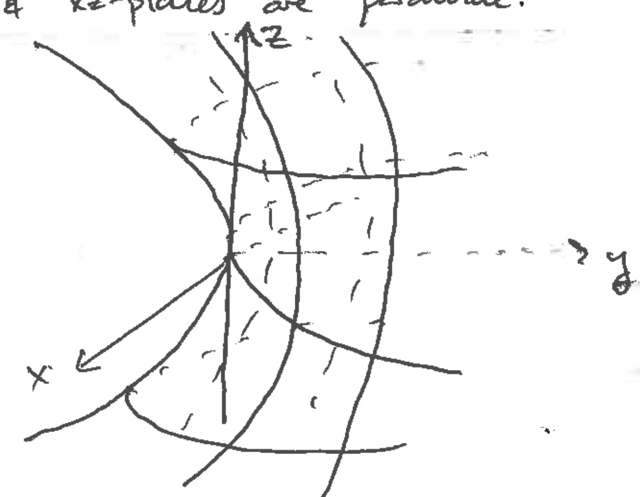
(e) $\frac{x^2}{4}+\frac{y^2}{9}-z=0$ $\longleftrightarrow$ $z=\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2$

Elliptic paraboloid along z-axis



(f) $\frac{x}{4}-\frac{y^2}{9}+z^2=0$ $\longleftrightarrow$ $x=\left(y/(3/2)\right)^2-\left(z/(1/2)\right)^2$

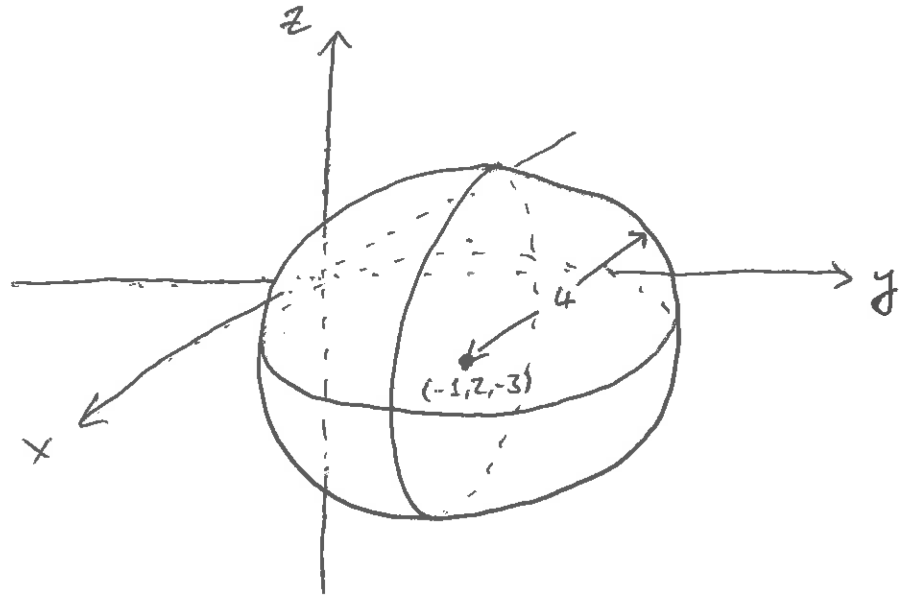
Hyperbolic paraboloid, intersections with (planes parallel to the) yz-plane are hyperbolae, intersections with planes parallel to the xy- & xz-planes are parabolae.



(g) $x^2+y^2+z^2+2x-4y+6z=2$

$\equiv (x+1)^2 + (y-2)^2 + (z+3)^2 = 16$

$\equiv \left(\frac{x+1}{4}\right)^2 + \left(\frac{y-2}{4}\right)^2 + \left(\frac{z+3}{4}\right)^2 = 1$ ⟵ Sphere of radius 4 centred at $(-1, 2, -3)$



(h) $(x,y,z)\cdot(y,z,x)=1$

$\equiv xy + yz + zx = 1 \equiv \vec{x}^{\,T} A \vec{x} = 1$ where $A = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Orthonormal eigenbasis of A:

$\mathcal{B} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

Eigenvalues: $\quad 1 \qquad -1/2 \qquad -1/2$

So in $\mathcal{B}$-coordinates, the surface is described by

$c_1^2 - \frac{1}{2}c_2^2 - \frac{1}{2}c_3^2 = 1$ ⟶ Hyperboloid of 2 sheets intersecting c_1-axis at ± 1.