

Math 290-2 Class 14

Monday 11th February 2019

Quadric surfaces

A *quadric surface* is a surface in \mathbb{R}^3 of the form $\mathbf{x}^T A \mathbf{x} + \mathbf{b} \cdot \mathbf{x} + c = 0$, where A is a 3×3 symmetric matrix, \mathbf{b} is a vector in \mathbb{R}^3 and c is a scalar. That is:

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + b_1x + b_2y + b_3z + c = 0$$

Some special cases include the following. [See supplemental handout for sketches.]

- Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Hyperboloid of two sheets: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Elliptic cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ — [think about what a ‘hyperbolic cone’ would be]
- Elliptic paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
- Hyperbolic paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Quadratic forms revisited (not in Colley)

Every quadratic form $q : \mathbb{R}^3 \rightarrow \mathbb{R}$ defines a quadric surface. If $q(x, y, z) = \mathbf{x}^T A \mathbf{x}$, with A symmetric, then we obtain

$$q(x, y, z) = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A and (c_1, c_2, c_3) are the coordinates of (x, y, z) with respect to the orthonormal eigenbasis $\vec{u}_1, \vec{u}_2, \vec{u}_3$.

We can then sketch $q(x, y, z) = 1$ on (c_1, c_2, c_3) -axes, just as we did for ellipses and hyperbolae in two dimensions. Note that a surface of the form $\mathbf{x}^T A \mathbf{x} = 1$ will either be an ellipsoid, a hyperboloid of one or two sheets, or an (elliptic or hyperbolic) *cylinder* (this occurs when one of the eigenvalues is zero).

1. Describe (and try to sketch) the following surfaces:

(a) $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$

(b) $\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$

(c) $\frac{x^2}{4} - \frac{y^2}{9} - z^2 = 1$

$$(d) \frac{x^2}{4} - \frac{y^2}{9} + z^2 = 1$$

$$(e) \frac{x^2}{4} + \frac{y^2}{9} - z = 0$$

$$(f) \frac{x}{4} - \frac{y^2}{9} + z^2 = 0$$

(g) $x^2 + y^2 + z^2 + 2x - 4y + 6z = 2$

(h) $(x, y, z) \cdot (y, z, x) = 1$