Math 290-2 Class 12

Wednesday 6th February 2019

Coordinate systems (repeated from Monday)

The **cartesian coordinates** (named after René Descartes) of a point P in $\mathbb{R}^n$ tell us how many units must be traversed from the origin in the direction of each standard basis vector to get to P. But this is not the only way of specifying a point—it may be more convenient to use a different coordinate system, such as one of those described below.

The **polar coordinates** of a point P in $\mathbb{R}^2$ are given by (r, θ), where:

- r is the distance of P from the origin; and
- θ is the angle of the ray from the origin on which P lies.

We may allow $r < 0$, in which case P lies $|r|$ units on the ray with angle $\theta + \pi$.

The conversion between cartesian and polar coordinates is given by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{and} \quad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \text{ or indeterminate if } x = 0 \end{cases}$$

The **cylindrical coordinates** of a point P in $\mathbb{R}^3$ are given by (r, θ, z), where:

- (r, θ) are the polar coordinates of the projection of P onto the (x, y)-plane; and
- z is the height of P along the z-axis.

The conversion between cartesian and cylindrical coordinates is given by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \text{and} \quad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \\ z = z \end{cases}$$

The **spherical coordinates** of a point P in $\mathbb{R}^3$ are given by (ρ, φ, θ), where:

- ρ is the distance of P from the origin;
- φ is the angle that $\overrightarrow{OP}$ makes with the positive z-axis; and
- θ is the angle of P about the z-axis.

The conversion between cartesian and spherical coordinates is given by

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases} \quad \text{and} \quad \begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan\varphi = \dfrac{\sqrt{x^2 + y^2}}{z} \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

1. Let C be the circle in the (x,z) plane with centre $(2,0,0)$ and radius 1. Find the cylindrical coordinate equation of the torus traced by C upon rotation by 2π radians about the z-axis.

2. Find the cartesian equation of the surface described in spherical coordinates by

$$\rho^2(a\sin^2\varphi\cos^2\theta + b\sin^2\varphi\sin^2\theta + c\cos^2\theta) = 1$$

Sketch this surface in when $a > b > c > 0$.

3. Let S be the cone whose cylindrical equation is $z = r$, $z \geqslant 0$. Describe S in spherical coordinates.

4. Find a way of converting between cylindrical and spherical coordinates.

5. Sketch the solid region of $\mathbb{R}^3$ described in spherical coordinates by

$$1 \leqslant \rho^2 \leqslant 4, \quad 0 \leqslant \varphi \leqslant \frac{\pi}{6}, \quad 0 \leqslant \theta \leqslant 2\pi$$

and describe this region in cartesian and cylindrical coordinates.