1. Find the parametric vector equation of the plane $2x-y+z=3$.

Let $x = s,\ y = t \rightsquigarrow z = 3-2s+t$

$$\rightsquigarrow \quad \vec{r}(s,t) = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

[Note Could have let $y = s$, $z = t$, etc., but then the expression for x would involve fractions — gross!]

2. Find the coordinate equation of the plane $\Pi : x = 2s-t,\ y = 1-s+t,\ z = 2-t$.

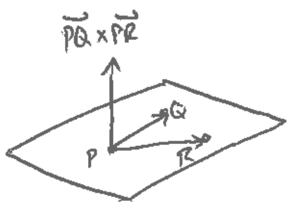
Vector equation: $\vec{r}(s,t) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

pt on plane   vectors parallel to plane

Normal vector: $\vec{n} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$$\rightsquigarrow \quad x + 2y + z = 1\cdot 0 + 2\cdot 1 + 1\cdot 2 = 4$$

3. Find the parametric vector equation and the coordinate equation of the plane passing through the points $P(1,0,1)$, $Q(1,-1,1)$ and $R(-2,1,2)$.



$$\vec{r}(t) = \underbrace{\overrightarrow{OP}}_{\text{pt on plane}} + \underbrace{s\overrightarrow{PQ} + t\overrightarrow{PR}}_{\text{vectors parallel to plane}} = \begin{pmatrix}1\\0\\1\end{pmatrix} + s\begin{pmatrix}0\\-1\\0\end{pmatrix} + t\begin{pmatrix}-3\\1\\1\end{pmatrix}$$ ← parametric vector equation

Normal vector: $\vec{n} = \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{pmatrix}0\\-1\\0\end{pmatrix}\times\begin{pmatrix}-3\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\-3\end{pmatrix}$

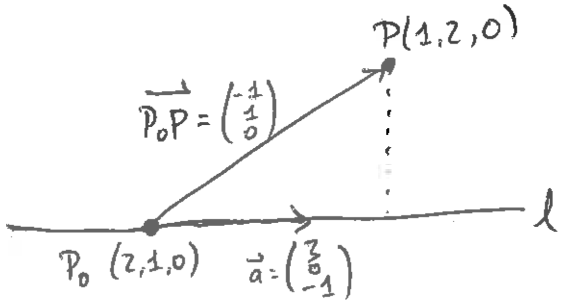
& $\begin{pmatrix}-1\\0\\-3\end{pmatrix}\cdot\underbrace{\begin{pmatrix}1\\0\\1\end{pmatrix}}_{\text{pt on plane}} = -4$

$\rightsquigarrow$ $-x - 3\cancel{y}z = -4$ (or $x + 3z = 4$)

4. Find the distance from the point $P(1,2,0)$ to the line $\ell:\ x = 2+3t,\ y = 1,\ z = -t$.

pt on line: $(2,1,0)$

direction vector: $(3,0,-1)$

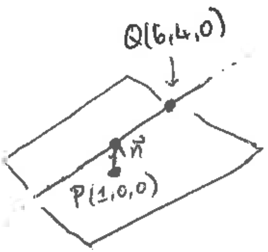


$$\text{dist}(P,\ell) = \frac{\left\|\begin{pmatrix}-1\\1\\0\end{pmatrix}\times\begin{pmatrix}3\\0\\-1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}3\\0\\-1\end{pmatrix}\right\|} = \frac{1}{\sqrt{10}}\left\|\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right\| = \frac{\sqrt{3}}{\sqrt{10}}$$

5. Let Π_1, Π_2, Π_3 be the pairwise intersecting planes in $\mathbb{R}^3$ defined by

$$\begin{aligned} \Pi_1 &: \ x - 2y + \ z = \ 1 \\ \Pi_2 &: 2x - 3y + \ z = \ 0 \\ \Pi_3 &: 3x - 5y + 2z = -2 \end{aligned}$$

Find the distance between Π_1 and the line of intersection of Π_2 and Π_3.



Pt on Π_1 : $P(1,0,0)$. Normal vector to Π_1: $\vec{n} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Pt on line of intersection:

$$\left(\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 3 & -5 & 2 & -2 \end{array}\right) \xrightarrow[R_2 \times 2]{R_1 \times 3} \left(\begin{array}{ccc|c} 6 & -9 & 3 & 0 \\ 6 & -10 & 4 & -4 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 6 & -9 & 3 & 0 \\ 0 & -1 & 1 & -4 \end{array}\right)$$

$$\downarrow R_2 \times (-1)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 6 \\ 0 & 1 & -1 & 4 \end{array}\right) \xleftarrow{R_1 \div 6} \left(\begin{array}{ccc|c} 6 & 0 & -6 & 36 \\ 0 & 1 & -1 & 4 \end{array}\right) \xleftarrow{R_1 + 9R_2} \left(\begin{array}{ccc|c} 6 & -9 & 3 & 0 \\ 0 & 1 & -1 & 4 \end{array}\right)$$

So take $x = 6$, $y = 4$, $z = 0 \Rightarrow Q(6,4,0)$ is on the line of intersection.

So $\text{dist}\left(\Pi_1, \begin{smallmatrix}\text{line of} \\ \text{intersection}\end{smallmatrix}\right) = \text{dist}\left(P, \begin{smallmatrix}\text{line of} \\ \text{intersection}\end{smallmatrix}\right) = \left| \dfrac{\vec{n} \cdot \vec{PQ}}{\|\vec{n}\|} \right|$

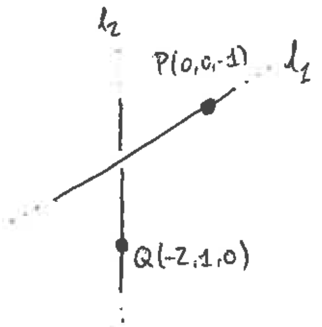
$$= \left| \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|} \right| = \left| \frac{1}{\sqrt{6}} (5 - 8 + 0) \right| = \left| \frac{-3}{\sqrt{6}} \right| = \frac{3}{\sqrt{6}}$$

6. Find the distance between the skew lines ℓ_1 and ℓ_2, defined by the parametric vector equations $\mathbf{r}(t) = (0,0,-1) + t(1,-3,2)$ and $\mathbf{r}(t) = (-2,1,0) + t(0,1,4)$, respectively.

$\text{dist}(\ell_1, \ell_2) = \text{dist}\left(\begin{smallmatrix}\text{point on} \\ \ell_1\end{smallmatrix}, \begin{smallmatrix}\text{plane parallel to both } \ell_1 \,\&\, \ell_2 \\ \text{which contains } \ell_2\end{smallmatrix} \right)$

$P(0,0,-1)$

pt on plane: $Q(-2,1,0)$



normal vector must be perp. to both $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$

$$\rightsquigarrow \vec{n} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -14 \\ -4 \\ 1 \end{pmatrix}$$

$$\Rightarrow \text{dist}(\ell_1, \ell_2) = \left| \frac{\vec{n} \cdot \vec{PQ}}{\|\vec{n}\|} \right| = \left| \frac{\begin{pmatrix} -14 \\ -4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}{\sqrt{196 + 16 + 1}} \right|$$

$$= \frac{1}{\sqrt{213}} |28 + 5 + 1| = \frac{34}{\sqrt{213}}$$