Math 290-2 Class 10

Friday 1st February 2019

Equations of planes

We can find the equation of a plane provided that we know (i) a point in the plane, and (ii) a normal vector to the plane. Indeed, if P_0 is a given point in the plane, and $\mathbf{n}$ is a normal (i.e. perpendicular) vector to the plane, then

$$P \text{ is in the plane} \quad \Leftrightarrow \quad \mathbf{n}\cdot\overrightarrow{P_0P} = 0$$

Now if $\mathbf{n} = (a,b,c)$ and $P_0 = (x_0,y_0,z_0)$, this says that

$$P(x,y,z) \text{ is in the plane} \quad \Leftrightarrow \quad \boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

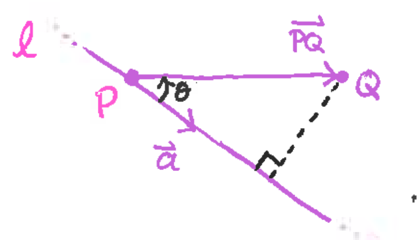
So we get an equation of the form $ax+by+cz=d$ by letting $d = ax_0+by_0+cz_0$. This is called the **coordinate equation** of the plane.

The equation $ax+by+cz=d$ is a system of 1 equation in 3 variables, so (provided a,b,c aren't all zero) its solutions take the form $\boxed{\mathbf{r}(s,t) = s\mathbf{a}+t\mathbf{b}+\mathbf{c}}$, where:

- $\mathbf{a}$ and $\mathbf{b}$ are two vectors parallel to the plane; and
- $\mathbf{c}$ is the coordinate vector of a point in the plane.

This is called the **parametric vector equation** of the plane.

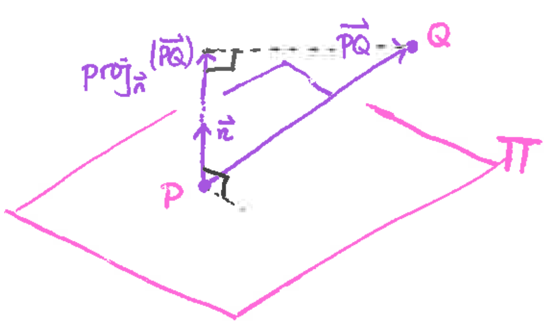
Distance between a point and a line



Consider a line ℓ passing through a point P and parallel to a vector $\mathbf{a}$. If Q is any other point, then the distance from Q to ℓ is $\|\overrightarrow{PQ}\|\sin\theta$, where θ is the angle between $\overrightarrow{PQ}$ and $\mathbf{a}$, and so

$$\operatorname{dist}(Q,\ell) = \frac{\|\overrightarrow{PQ}\|\,\|\mathbf{a}\|\sin\theta}{\|\mathbf{a}\|} = \boxed{\frac{\|\overrightarrow{PQ}\times\mathbf{a}\|}{\|\mathbf{a}\|}}$$

Distances between a point and a plane



Consider a plane Π passing through a point P and perpendicular to a vector $\mathbf{n}$. If Q is any other point, then the distance from Q to Π is the length of the orthogonal projection of $\overrightarrow{PQ}$ onto $\mathbf{n}$, and so

$$\operatorname{dist}(Q,\Pi) = \|\operatorname{proj}_{\mathbf{n}}(\overrightarrow{PQ})\| = \left\|\left(\frac{\mathbf{n}\cdot\overrightarrow{PQ}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}\right\| = \boxed{\frac{|\mathbf{n}\cdot\overrightarrow{PQ}|}{\|\mathbf{n}\|}}$$

If $\mathbf{n}$ is a unit vector then this simplifies to $\operatorname{dist}(Q,\Pi) = |\mathbf{n}\cdot\overrightarrow{PQ}|$.

1. Find the parametric vector equation of the plane $2x-y+z=3$.

2. Find the coordinate equation of the plane $\Pi:\ x=2s-t,\ y=1-s+t,\ z=2-t$.

3. Find the parametric vector equation and the coordinate equation of the plane passing through the points $P(1,0,1)$, $Q(1,-1,1)$ and $R(-2,1,2)$.

4. Find the distance from the point $P(1,2,0)$ to the line $\ell:\ x=2+3t,\ y=1,\ z=-t$.

5. Let Π_1, Π_2, Π_3 be the pairwise intersecting planes in $\mathbb{R}^3$ defined by

$$\begin{aligned} \Pi_1 &: \; x - 2y + z = 1 \\ \Pi_2 &: 2x - 3y + z = 0 \\ \Pi_3 &: 3x - 5y + 2z = -2 \end{aligned}$$

Find the distance between Π_1 and the line of intersection of Π_2 and Π_3.

6. Find the distance between the skew lines ℓ_1 and ℓ_2, defined by the parametric vector equations $\mathbf{r}(t) = (0,0,-1) + t(1,-3,2)$ and $\mathbf{r}(t) = (-2,1,0) + t(0,1,4)$, respectively.