Math 290-2 Class 9

Monday 28th January 2019

The cross product

Recall that the **dot product** (a.k.a. *scalar product*) $\mathbf{a}\cdot\mathbf{b}$ of two vectors in $\mathbb{R}^n$ is a *scalar*, defined by $a_1b_1+a_2b_2+\cdots+a_nb_n$. The **cross product** (a.k.a. *vector product*) $\mathbf{a}\times\mathbf{b}$ of two vectors is a *vector* such that:

(i) $\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$; its direction is determined by the 'right hand rule'.

(ii) $\|\mathbf{a}\times\mathbf{b}\|$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

For example, in $\mathbb{R}^3$, we must have:

$$\begin{array}{lll} \mathbf{i}\times\mathbf{j}=\ \ \mathbf{k} & \mathbf{j}\times\mathbf{k}=\ \ \mathbf{i} & \mathbf{k}\times\mathbf{i}=\ \ \mathbf{j} \\ \mathbf{j}\times\mathbf{i}=-\mathbf{k} & \mathbf{k}\times\mathbf{j}=-\mathbf{i} & \mathbf{i}\times\mathbf{k}=-\mathbf{j} \end{array}$$

Some arithmetic properties of the cross product include:

- (Anticommutativity) $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$
- (Distributivity) $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c})$ and $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})$;
- (Multilinearity) $k(\mathbf{a}\times\mathbf{b})=(k\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(k\mathbf{b})$;

Some more cross product fun facts:

- $\mathbf{a}$ is parallel to $\mathbf{b}\Leftrightarrow\mathbf{a}\times\mathbf{b}=\mathbf{0}$.
- If the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ (where $0\leqslant\theta<\pi$), then $\|\mathbf{a}\times\mathbf{b}\|=\|\mathbf{a}\|\|\mathbf{b}\|\sin(\theta)$.
- Cross products can only be defined in $\mathbb{R}^3$ and $\mathbb{R}^7$; we will only study the cross product in $\mathbb{R}^3$. [There are 480 different cross products in $\mathbb{R}^7$—eek!]

We can derive the following formula for vectors in $\mathbb{R}^3$.

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix} a_2b_3-a_3b_2 \\ a_3b_1-a_1b_3 \\ a_1b_2-a_2b_1 \end{pmatrix}=\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

The **scalar triple product** of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is the (scalar!) quantity $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. Fun fact:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

and so $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$ is the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Note also that

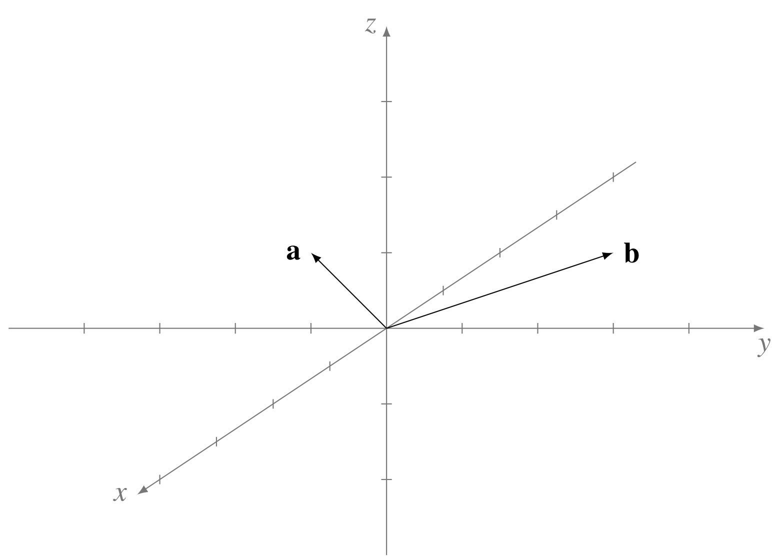
$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$$

and that $\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b})=-\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, and so on.

1. Let **a** and **b** be two vectors in $\mathbb{R}^3$. Expand the following expression and simplify it as much as possible.

$$(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b})$$

2. Sketch $\mathbf{a} \times \mathbf{b}$ in the following diagram, where **a** and **b** are in the (y,z)-plane.



3. Evaluate $(2,1,-1)\times(1,0,3)$.

4. Find the equation of the plane that passes through the point $P(-1,3,2)$ and is parallel to the vectors $\mathbf{a}=2\mathbf{i}-\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-2\mathbf{j}$.

5. For each of the following statements about vectors **a**, **b**, and **c** in $\mathbb{R}^3$, determine whether it is always, sometimes or never true.

(a) $\|\mathbf{a}\times\mathbf{b}\|^2+(\mathbf{a}\cdot\mathbf{b})^2=\|\mathbf{a}\|^2\|\mathbf{b}\|^2$

(b) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$

(c) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=\mathbf{0}$

(d) Suppose $\mathbf{a},\mathbf{b},\mathbf{c}$ is an orthonormal basis of $\mathbb{R}^3$. Then $\mathbf{a}\times\mathbf{b}=\mathbf{c}$.