Math 290-2 Class 8

Friday 25th January 2019
(Burns Night)

Ance mair I hail thee, thou gloomy January!
Ance mair I hail thee wi' sorrow and care;
Sad was the parting thou makes me remember—
Parting wi' Bretscher, oh, ne'er to meet mair!
— Robert Burns, 1781 (modified)

Comments on the new book

We are now using *Vector Calculus* (4th ed.) by Susan J. Colley (ISBN 978-0-321-78065-2). A couple of conventions used are slightly different from those in Bretscher's book:

- In ($\mathbb{R}^2$ and) $\mathbb{R}^3$, the standard basis vectors are denoted by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ rather than $\vec{e}_1, \vec{e}_2, \vec{e}_3$;
- The book uses the notation (x, y, z) to mean both a point P in $\mathbb{R}^3$ *and* its position vector $\overrightarrow{OP}$ (except briefly in Chapter 2 when row and column vectors are introduced). In Bretscher, the vector $\overrightarrow{OP}$ would be written as a *column vector*. The distinction between points and position vectors is subtle and won't cause too many problems, but keep it in the back of your mind.

In the spirit of turning over a new leaf, we will also transition to using Colley's notation for vectors, and we will begin to blur the distinction between points and their position vectors (making it explicit where necessary).

Parametric equations

Consider the equation $ax + by = c$ in $\mathbb{R}^2$. It describes a line ℓ, where a point (x, y) on ℓ is exactly a point whose coordinates satisfy the equation.

If $c = 0$, we could also describe ℓ as $\operatorname{span}\{(-b, a)\}$. The points on ℓ are therefore those of the form $(-bt, at)$ as t varies over $\mathbb{R}$. But what if $c \neq 0$? In that case, if we knew that (x_0, y_0) were a point on ℓ, then $(x, y) = (x_0 - bt, y_0 + at)$ would be on ℓ, since

$$ax + by = a(x_0 - bt) + b(y_0 + at) = (ax_0 + by_0) + (-abt + abt) = c + 0 = c$$

So we can express ℓ to be the set of points $(x(t), y(t))$ of the form

$$(x(t), y(t)) = (x_0 - bt, y_0 + at) = (x_0, y_0) + t(-b, a)$$

Thus a line ℓ that passes through a point $\mathbf{b}$ and is parallel to the vector $\mathbf{a}$ can be described by

$$\boxed{\mathbf{r}(t) = \mathbf{b} + t\mathbf{a}}$$

This is called the *vector equation* of ℓ. Notice that this works in any dimension, not just two.

More generally, vectors $\mathbf{r}(t)$ describe *curves* in $\mathbb{R}^n$. The varaible t is called a *parameter*. [Later, we will increase the number of parameters, which will allow us to describe surfaces and higher-dimensional solids.]

1. Find the vector equation of the line $2x+3y=1$ in $\mathbb{R}^2$.

2. Find the vector equation of the line of intersection of the planes in $\mathbb{R}^3$ given by

$$x+y-z=-1 \quad \text{and} \quad x+2y-2z=1$$

3. Show that every point (x,y,z) on the line in $\mathbb{R}^3$ with vector equation $\mathbf{r}(t)=\mathbf{b}+t\mathbf{a}$ satisfies

$$\frac{x-b_1}{a_1}=\frac{x-b_2}{a_2}=\frac{x-b_3}{a_3}$$

as long as $a_1,a_2,a_3\neq 0$. This is called the **symmetric form** of the line.

4. Sketch the curve in $\mathbb{R}^2$ parametrised by $\mathbf{r}(t) = (\cos(t), \sin(t))$.

5. Sketch the curve in $\mathbb{R}^2$ parametrised by $\mathbf{r}(t) = (t\cos t, t\sin t)$ for $t \geqslant 0$.

6. Sketch the curve in $\mathbb{R}^3$ parametrised by $\mathbf{r}(t) = (\cos t, \sin t, t)$.