Math 290-2 Class 7

Monday 22nd January 2019

Orthogonal change-of-basis and conic sections

Recall that if $\mathfrak{B}$ is a basis of $\mathbb{R}^n$, then the transition matrix S, whose columns are the vectors in $\mathfrak{B}$, allows us to translate between standard coordinates and $\mathfrak{B}$-coordinates: applying S 'decodes' $\mathfrak{B}$-coordinates ($S[\vec{x}]_{\mathfrak{B}} = \vec{x}$), and applying S^{-1} 'encodes' into $\mathfrak{B}$-coordinates ($S^{-1}\vec{x} = [\vec{x}]_{\mathfrak{B}}$).

When the vectors in $\mathfrak{B}$ are orthonormal, the matrix S is orthogonal, and so angles and lengths are preserved when we view the world through the lens of $\mathfrak{B}$-coordinates.

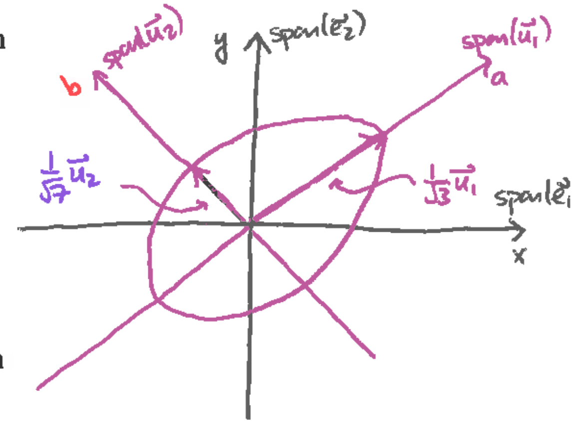
For example, let $\mathfrak{B} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$, and consider the equation

$$5x^2 - 4xy + 5y^2 = 1$$

Letting $\begin{pmatrix}a\\b\end{pmatrix}$ be the $\mathfrak{B}$-coordinate vector of $\begin{pmatrix}x\\y\end{pmatrix}$, it turns out that

$$3a^2 + 7b^2 = 1$$

We see from the point of view of $\mathfrak{B}$-coordinates that the equation describes an ellipse, whose principal axes are the vectors in $\mathfrak{B}$.



Quadratic forms

A **quadratic form** is a function $q : \mathbb{R}^n \to \mathbb{R}$ such that $q(\vec{x})$ is a linear combination of terms of the form $x_i x_j$. For example, in $\mathbb{R}^2$, all quadratic forms take the form $q(x,y) = ax^2 + bxy + cy^2$.

Every quadratic form can be expressed (uniquely!) as $q(\vec{x}) = \vec{x}^T A \vec{x}$ for some symmetric matrix A. For example

$$ax^2 + by^2 + cz^2 + pxy + qxz + ryz = \begin{pmatrix}x\\y\\z\end{pmatrix}^T \begin{pmatrix}a & p/2 & q/2\\ p/2 & b & r/2\\ q/2 & r/2 & c\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Let $\mathfrak{B}$ is an orthonormal eigenbasis of A, whose respective eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_n$, and let S be the (orthogonal) transition matrix of $\mathfrak{B}$. For any vector $\vec{x}$, let $\vec{c}$ be its $\mathfrak{B}$-coordinate vector and let $D = S^T A S$ be the diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$. Then

$$q(\vec{x}) = \vec{x}^T A \vec{x} = (S\vec{c})^T A (S\vec{c}) = \vec{c}^T S^T A S \vec{c} = \vec{c}^T D \vec{c} = \boxed{\lambda_1 c_1^2 + \lambda_2 c_2^2 + \cdots + \lambda_n c_n^2}$$

This gives us very useful information about the quadratic form and about A.

Definiteness

Let A be a symmetric $n \times n$ matrix and let $q(\vec{x}) = \vec{x}^T A\vec{x}$ be its associated quadratic form. Then

- We say A is **positive definite** if $q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$, and **positive semidefinite** if $q(\vec{x}) \geqslant 0$ for all $\vec{x}$.
- We say A is **negative definite** if $q(\vec{x}) < 0$ for all $\vec{x} \neq \vec{0}$, and **negative semidefinite** if $q(\vec{x}) \leqslant 0$ for all $\vec{x}$.
- We say A is **indefinite** if $q(\vec{x})$ takes both positive and negative values.

Knowing the definiteness of a symmetric matrix allows us to reason about whether its associated quadratic form has global maxima and minima—when we study vector calculus, this will allow us to classify local extrema of surfaces described by multivalued functions.

Some useful facts:

- A is positive definite $\Leftrightarrow$ all its eigenvalues are positive, and A is positive semidefinite $\Leftrightarrow$ all its eigenvalues are nonnegative. [Likewise for negative (semi)definiteness.]
- A is positive definite $\Leftrightarrow$ $\det(A^{(k)}) > 0$ for all $1 \leqslant k \leqslant n$, where $A^{(k)}$ is the top left $k \times k$ submatrix of A (called a **principal submatrix**).

It turns out that the definiteness of a matrix gives us useful information about surfaces of the form $ax^2 + bxy + cy^2 = 1$ in $\mathbb{R}^2$. Indeed, let $A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ be the associated symmetric matrix. Then

- $ax^2 + bxy + cy^2 = 1$ describes an ellipse if and only if A is positive definite;
- $ax^2 + bxy + cy^2 = 1$ describes a hyperbola if and only if A is indefinite;
- In both cases, the principal axes of the curve in question are given by the eigenspaces of A.

More generally, the principal axes of a quadratic form $q(\vec{x}) = \vec{x}^T A\vec{x}$, where A is a symmetric matrix with distinct eigenvalues, are the eigenspaces of A.

1. For each of the following quadratic forms, find a matrix A such that $q(\vec{x}) = \vec{x}^T A\vec{x}$.

(a) $q(x,y) = x^2 - 2xy - 4y^2$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & -4 \end{pmatrix}$$

(b) $q(x,y,z) = -2x^2 + y^2 - 3z^2 - 4xy - 6xz + 8yz$

$$A = \begin{pmatrix} -2 & -2 & -3 \\ -2 & 1 & 4 \\ -3 & 4 & -3 \end{pmatrix}$$

2. [*Bretscher §8.2 Ex 1, modified*]
Find the global maxima and minima of the ~~quadratic form~~ function ~~q~~ f: $\mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = 8x^2 + 5y^2 - 4xy + 1$$

Let $q(x,y) = 8x^2 + 5y^2 - 4xy = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$

$f_A(\lambda) = \lambda^2 - 13\lambda + 36 = (\lambda - 4)(\lambda - 9)$

$\Rightarrow q(x,y) = 4c_1^2 + 9c_2^2$ where $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is the coordinate vector of $\begin{pmatrix} x \\ y \end{pmatrix}$ wrt an orthonormal eigenbasis of A.

$\Rightarrow q(x,y) \geq 0$ for all (x,y) & $q(0,0) = 0$

$\Rightarrow q$ has a global minimum value of 0

$\Rightarrow f$ has a global minimum value of 1 at $(0,0)$.

(& q, and hence f, are both unbounded
$\Rightarrow$ no global maxima)

3. For each of the following symmetric matrices, determine whether it is positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite.

(a) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$\det(A^{(1)}) = 1, \quad \det(A^{(2)}) = 0$

$\Rightarrow$ positive semidefinite

(b) $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

$\det(A^{(1)}) = 3, \quad \det(A^{(2)}) = 9 - 4 = 5$

$\Rightarrow$ positive definite

(c) $\begin{pmatrix} 9 & -1 & 2 \\ -1 & 7 & -3 \\ 2 & -3 & 3 \end{pmatrix}$ [*taken from Bretscher* §8.2 *Ex 2*]

$\det(A^{(1)}) = 9, \quad \det(A^{(2)}) = 9 \times 7 - 1 = 62$

$\det(A^{(3)}) = 2 \begin{vmatrix} -1 & 2 \\ 7 & -3 \end{vmatrix} - 3 \begin{vmatrix} 9 & 2 \\ -1 & -3 \end{vmatrix} + 3 \underbrace{\begin{vmatrix} 9 & -1 \\ -1 & 7 \end{vmatrix}}_{=62}$

$= 2(3 - 14) - 3(-27 + 2) + 3 \cdot 62$

$= 2 \cdot (-11) - 3 \cdot (-25) + 3 \cdot 62$

$= -22 + 75 + 3 \cdot 62$

> 0

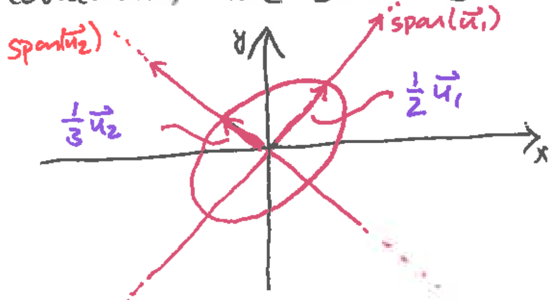
$\Rightarrow$ $\boxed{\text{+ve definite}}$

4. Sketch the curve in $\mathbb{R}^2$ defined by $8x^2 - 4xy + 5y^2 = 1$.

- $q(x,y) = 8x^2 - 4xy + 5y^2 = \vec{x}^T A \vec{x}$ where $A = \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$
- $\det(A^{(1)}) = 8$, $\det(A^{(2)}) = 40 - 4 = 36 \Rightarrow A$ is +ve definite so the curve is an ellipse.
- $f_A(\lambda) = \lambda^2 - 13\lambda + 36 = (\lambda - 4)(\lambda - 9) \Rightarrow$ evals are $4, 9$
- $A - 4I = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector
- $A - 9I = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is an eigenvector

$\Rightarrow \mathcal{B} = \underbrace{\tfrac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\vec{u}_1}, \underbrace{\tfrac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{\vec{u}_2}$

So in $\mathcal{B}$-coordinates, the curve is $4c_1^2 + 9c_2^2 = 1$



5. Sketch the curve in $\mathbb{R}^2$ defined by $3x^2 + 8xy - 3y^2 = 1$.

- $q(x,y) = 3x^2 + 8xy - 3y^2 = \vec{x}^T A \vec{x}$ where $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$
- $\det(A^{(1)}) = 3$, $\det(A^{(2)}) = -25 \Rightarrow A$ is indefinite so the curve is a hyperbola
- $f_A(\lambda) = \lambda^2 - 0\lambda - 25 = (\lambda - 5)(\lambda + 5) \Rightarrow$ e vals are $5, -5$
- $A - 5I = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector
- $A + 5I = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector

$\Rightarrow \mathcal{B} = \tfrac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tfrac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

So in $\mathcal{B}$-coordinates the curve is $5c_1^2 - 5c_2^2 = 1$