Math 290-2 Class 6

Friday 18th January 2019

Orthonormal bases of eigenvectors

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation given by $T(\vec{x}) = A\vec{x}$, and let $\mathfrak{B} = \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a basis of $\mathbb{R}^n$ with transition matrix S (that is, the columns of S are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$).

- We're happy when $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are eigenvectors of T, since then $S^{-1}AS$ is diagonal.
- We're happy when $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are orthonormal, since then S is orthogonal (so $S^T = S^{-1}$) and finding $\mathfrak{B}$-coordinate vectors is made much easier.

So if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are eigenvectors of T *and* they're orthonormal, then we won't be able to wipe the smiles off our faces, because then S^TAS is diagonal *and* working with $\mathfrak{B}$-coordinates is easy.

The question is… under what conditions does T have an orthonormal basis of eigenvectors (that is, when is A *orthogonally diagonalisable*)? Well:

- If A is orthogonally diagonalisable, then S is orthogonal, so $D = S^TAS$ is diagonal. (Notice also that $D^T = D$.) But then using the fact that $(PQ)^T = Q^TP^T$ we have

$$A^T = (SDS^T)^T = (S^T)^TD^TS^T = SD^TS^T = SDS^T = A$$

- If $A^T = A$, then A is orthogonally diagonalisable. (Proofs overleaf.)

A square matrix A such that $A^T = A$ is called a **symmetric** matrix. So we have:

$$\boxed{A \text{ is symmetric} \quad \Leftrightarrow \quad A \text{ is orthogonally diagonalisable}}$$

This result is called the **spectral theorem**. Other fun facts about symmetric matrices:

- A square matrix A is symmetric if and only if $(A\vec{v}) \cdot \vec{w} = \vec{v} \cdot (A\vec{w})$ for all vectors $\vec{v}$ and $\vec{w}$;
- If $\vec{v}$ and $\vec{w}$ are eigenvectors of a symmetric matrix A with distinct eigenvalues, then $\vec{v}$ and $\vec{w}$ are perpendicular;
- The characteristic polynomial $f_A(\lambda) = \det(A - \lambda I)$ of any symmetric matrix A can be completely factorised over the real numbers—consequently, A is diagonalisable and all its eigenvalues are real.

We can find an orthogonal matrix S such that $S^{-1}AS = S^TAS$ is diagonal as follows:

Step 1 Solve $f_A(\lambda) = 0$ to find the eigenvalues of A.

Step 2 For each eigenvalue λ, find a basis of E_λ and then orthonormalise it using Gram–Schmidt.

Step 3 Put these bases together to obtain an orthonormal basis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ of eigenvectors of A. These form the columns of an orthogonal matrix S such that S^TAS is diagonal.

Proofs

- If A is symmetric if and only if $(A\vec{v})\cdot\vec{w} = \vec{v}\cdot(A\vec{w})$ for all vectors $\vec{v}$ and $\vec{w}$.

  Proof. Suppose $A^T = A$. Then

  $$(A\vec{v})\cdot\vec{w} = (A\vec{v})^T\vec{w} = \vec{v}^T A^T \vec{w} = \vec{v}^T A\vec{w} = \vec{v}\cdot(A\vec{w})$$

  as required.

  Conversely, suppose $(A\vec{v})\cdot\vec{w} = \vec{v}\cdot(A\vec{w})$ for all vectors $\vec{v}$ and $\vec{w}$. Then $A\vec{e}_i$ is the i^{th} column of A, and so $(A\vec{e}_i)\cdot\vec{e}_j$ is the $(i,j)^{\text{th}}$ entry a_{ij} of A. Thus we have

  $$a_{ij} = (A\vec{e}_i)\cdot\vec{e}_j = \vec{e}_i\cdot(A\vec{e}_j) = (A\vec{e}_j)\cdot\vec{e}_i = a_{ji}$$

  But a_{ji} is the $(i,j)^{\text{th}}$ entry of A^T. Since A and A^T have the same number in each entry, it follows that $A^T = A$. □

- If $\vec{v}$ and $\vec{w}$ are distinct eigenvectors of a symmetric matrix A, then $\vec{v}$ is perpendicular to $\vec{w}$.

  Proof. Let λ be the eigenvalue of $\vec{v}$ and μ be the eigenvalue of $\vec{w}$. Then

  $$(\lambda-\mu)\vec{v}\cdot\vec{w} = (\lambda\vec{v})\cdot\vec{w} - \vec{v}\cdot(\mu\vec{w}) = (A\vec{v})\cdot\vec{w} - \vec{v}\cdot(A\vec{w}) = \vec{v}\cdot(A\vec{w}) - \vec{v}\cdot(A\vec{w}) = 0$$

  Since $\lambda-\mu \neq 0$, we must have $\vec{v}\cdot\vec{w} = 0$. □

- If λ is an eigenvalue of a symmetric matrix A, then λ is real.

  Proof. Let $\vec{v}$ be a complex eigenvector of A with complex eigenvalue λ. Since A has real entries, we have $A\vec{v}^* = (A\vec{v})^* = (\lambda\vec{v})^* = \lambda^*\vec{v}^*$, so $\vec{v}^*$ is an eigenvector with eigenvalue λ^*. Now we have

  $$(\lambda-\lambda^*)(\vec{v}\cdot\vec{v}^*) = (\lambda\vec{v})\cdot\vec{v}^* - \vec{v}\cdot(\lambda^*\vec{v}^*) = (A\vec{v})\cdot\vec{v}^* - \vec{v}\cdot(A\vec{v}^*) = \vec{v}\cdot(A\vec{v}^*) - \vec{v}\cdot(A\vec{v}^*) = 0$$

  But $\vec{v}\cdot\vec{v}^* = \|\vec{v}\|^2 \neq 0$, so we must have $\lambda-\lambda^* = 0$, so that λ is real. □

- The characteristic polynomial $f_A(\lambda)$ of a symmetric matrix A can be completely factorised.

  Proof. By the fundamental theorem of algebra, $f_A(\lambda)$ splits into linear factors of the form $\lambda-\lambda_i$. By what we just proved, each λ_i is real. □

- If A is symmetric, then A is orthogonally diagonalisable.

  Proof. Since $f_A(\lambda)$ splits into linear factors, A is diagonalisable, so it has a basis of eigenvectors. Each eigenspace has an orthonormal basis by the Gram–Schmidt process, and vectors in each eigenspace are perpendicular to each other by what we proved above, so these form an orthonormal basis of $\mathbb{R}^n$. □

1. For each of the following statements, determine whether it is always, sometimes or never true.

(a) The matrix of orthogonal projection onto a subspace of $\mathbb{R}^n$ is symmetric.

(b) Let a, b and c be real numbers. Then $\begin{pmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{pmatrix}$ is orthogonally diagonalisable.

(c) Let A be a symmetric matrix. Then A is invertible.

(d) Let A be a matrix such that if $\vec{v}$ and $\vec{w}$ are eigenvectors of A with distinct eigenvalues, then $\vec{v}$ is perpendicular to $\vec{w}$. Then A is orthogonally diagonalisable.

(e) Let A be the matrix of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by rotation by θ radians about the line spanned by $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, where $0 < \theta < \pi$. Then A is symmetric.

2. Find an orthogonal matrix S such that $S^T AS$ is diagonal, where $A = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$.

3. Let A be a symmetric, orthogonal 2×2 matrix. Show that either $A = \pm I_2$, or A is a reflection.

[Recall that a 2×2 reflection matrix is one of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ where $a^2 + b^2 = 1$.]

4. Let k be a real number. Find an orthonormal basis of eigenvectors of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\vec{x}) = A\vec{x}$, where A is the 3×3 matrix defined in terms of k by

$$A = \begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}$$