Math 290-2 Class 4

Monday 14th January 2019

Orthogonal transformations

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is *orthogonal* if T preserves dot products—more precisely, if $T(\vec{x}) \cdot T(\vec{y}) = \vec{x} \cdot \vec{y}$ for all vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^n$. Some fun facts about orthogonal transformations:

- T is orthogonal $\Leftrightarrow$ T preserves lengths—that is, $\|T(\vec{x})\| = \|\vec{x}\|$ for all vectors $\vec{x}$ in $\mathbb{R}^n$.
- If T is orthogonal and $\vec{x}$ and $\vec{y}$ are perpendicular, then $T(\vec{x})$ and $T(\vec{y})$ are perpendicular.
- T is orthogonal $\Leftrightarrow$ $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ are an orthonormal basis of $\mathbb{R}^n$.

Transposes and orthogonal matrices

Recall that the *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix A^T, where the $(i, j)^{\text{th}}$ component of A^T is the $(j, i)^{\text{th}}$ component of A:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

Transposes satisfy some nice properties, like $(AB)^T = B^T A^T$ and $(A^T)^{-1} = (A^{-1})^T$. Suppose the columns of A are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Then

$$A^T A = \begin{pmatrix} \cdots & \vec{v}_1^T & \cdots \\ \cdots & \vec{v}_2^T & \cdots \\ & \vdots & \\ \cdots & \vec{v}_n^T & \cdots \end{pmatrix} \begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \vdots & \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \cdots & \vec{v}_1 \cdot \vec{v}_n \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 & \cdots & \vec{v}_2 \cdot \vec{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_n \cdot \vec{v}_1 & \vec{v}_n \cdot \vec{v}_2 & \cdots & \vec{v}_n \cdot \vec{v}_n \end{pmatrix}$$

A matrix A is orthogonal if the linear transformation $T(\vec{x}) = A\vec{x}$ is an orthogonal transformation. But T is orthogonal if and only if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are orthonormal, since $\vec{v}_i = T(\vec{e}_i)$ for each $1 \leqslant i \leqslant n$. Therefore

$$\boxed{A \text{ is orthogonal} \quad \Leftrightarrow \quad A^T A = I_n}$$

since $\vec{v}_i \cdot \vec{v}_j = 1$ if $i = j$ and 0 if $i \neq j$. In particular, all orthogonal matrices are invertible.

Orthogonal projections... again

If V has orthonormal basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$, then the matrix of orthogonal projection onto V is given by QQ^T (not $Q^T Q$!!), where Q is the $n \times k$ matrix whose columns are $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$.

Proofs

This section contains (relatively short) proofs of the results overleaf. You do not need to know these by heart, but you should be able to understand them.

- T is orthogonal $\Leftrightarrow$ T preserves lengths.

  Proof. This is because the dot product can be expressed in terms of the length and vice versa:
  $$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} \quad \text{and} \quad \vec{a}\cdot\vec{b} = \|\vec{a}+\vec{b}\|^2 - \|\vec{a}\|^2 - \|\vec{b}\|^2$$
  So if T preserves one, then T preserves the other.

- If T is orthogonal, then T preserves angles.

  Proof. If the angle between $\vec{x}$ and $\vec{y}$ is θ, then
  $$\theta = \arccos\left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\|\vec{y}\|}\right) = \arccos\left(\frac{T(\vec{x})\cdot T(\vec{y})}{\|T(\vec{x})\|\|T(\vec{y})\|}\right)$$
  so the angle between $T(\vec{x})$ and $T(\vec{y})$ is also θ.

- If T is orthogonal and $\vec{x}$ and $\vec{y}$ are perpendicular, then $T(\vec{x})$ and $T(\vec{y})$ are perpendicular.

  Proof. This follows from the fact that T preserves angles, with $\theta = \frac{\pi}{2}$.

- If T is orthogonal, then $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ are an orthonormal basis of $\mathbb{R}^n$.

  Proof. Since the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are orthonormal and T preserves lengths and angles, the vectors $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ are also orthonormal.

- If $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ are an orthonormal basis of $\mathbb{R}^n$, then T is orthogonal.

  Proof. Write $\vec{u}_i = T(\vec{e}_i)$ for each $1 \leqslant i \leqslant n$, so that $T(\vec{x}) = x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_n\vec{u}_n$ for each $\vec{x}$ in $\mathbb{R}^n$. Then
  $$\begin{aligned}
  T(\vec{x})\cdot T(\vec{y}) &= \left(\sum_{i=1}^{n} x_i\vec{u}_i\right)\left(\sum_{j=1}^{n} y_j\vec{u}_j\right) && \text{by definition of dot product}\\
  &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j(\vec{u}_i\cdot\vec{u}_j) && \text{by linearity}\\
  &= \sum_{i=1}^{n} x_i y_i && \text{since } \vec{u}_i\cdot\vec{u}_j = 0 \text{ if } i \neq j \text{ and } 1 \text{ if } i = j\\
  &= \vec{x}\cdot\vec{y} && \text{by definition of dot product}
  \end{aligned}$$
  Hence T is orthogonal.

- If $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$ is an orthonormal basis of a subspace V of $\mathbb{R}^n$, then the matrix of proj_V is QQ^T, where Q is the $n\times k$ matrix with columns $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$.

  Proof. First notice that for all $1 \leqslant j \leqslant k$ we have $Q^T\vec{u}_j = \vec{e}_j$, and that if $\vec{w}$ is perpendicular to V then $Q^T\vec{w} = \vec{0}$. Extend the orthonormal basis of V to an orthonormal basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ of $\mathbb{R}^n$. If $j \leqslant k$ then $QQ^T\vec{u}_j = Q\vec{e}_j = \vec{u}_j = \text{proj}_V(\vec{u}_j)$, and if $j > k$ then $QQ^T\vec{u}_j = Q\vec{0} = \vec{0} = \text{proj}_V(\vec{u}_j)$. Since QQ^T and proj_V agree on a basis, we have $QQ^T(\vec{x}) = \text{proj}_V(\vec{x})$ for all $\vec{x}$ in $\mathbb{R}^n$.

1. (a) Show that the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

(b) Suppose that $a^2+b^2=1$. Show that the matrix $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ is orthogonal.

(c) (Try at home:) Show that all orthogonal 2×2 matrices are of the form (a) or (b). Hence all orthogonal transformations $\mathbb{R}^2 \to \mathbb{R}^2$ are rotations and reflections.

2. For each of the following statements, determine whether it is always, sometimes or never true.

(a) Let A be an orthogonal matrix. Then $\det(A) = \pm 1$.

(b) Let A be a 2×3 matrix. Then $A^T A$ is orthogonal.

(c) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. If the angle between vectors $\vec{x}$ and $\vec{y}$ is θ, then the angle between $T(\vec{x})$ and $T(\vec{y})$ is θ.

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(d) Suppose $\det(A) = 1$. Then A is orthogonal.

(e) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\0\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Then T is orthogonal.

3. (a) Find an orthonormal basis of the plane V in $\mathbb{R}^3$ described by the equation $2x + y - 3z = 0$.

(b) Find the matrix of orthogonal projection onto V.