Math 290-2 Class 3

Friday 11th January 2019

Gram–Schmidt orthonormalisation

We like orthonormal bases, but alas, not all bases are created orthonormal. The *Gram–Schmidt process* is a recursive procedure for turning a basis of (a subspace of) $\mathbb{R}^n$ into an orthonormal basis (of the same subspace).

The procedure goes like this. Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ be linearly independent vectors in $\mathbb{R}^n$.

Step 1. Divide $\vec{v}_1$ by its length to obtain a unit vector $\vec{u}_1$ parallel to $\vec{v}_1$:

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1$$

Step 2. Define $\vec{v}_2^{\perp} = \vec{v}_2 - \mathrm{proj}_{\vec{v}_1}(\vec{v}_2)$, and observe:

(a) $\vec{v}_2^{\perp}$ is perpendicular to $\vec{v}_1$; and

(b) $\vec{v}_2^{\perp}$ is in the span of $\vec{v}_1$ and $\vec{v}_2$;

And then divide $\vec{v}_2^{\perp}$ by its length to obtain a unit vector $\vec{u}_2$ parallel to $\vec{v}_2^{\perp}$:

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 \qquad \text{and} \qquad \vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp}$$

Step 3. Define $\vec{v}_3^{\perp} = \vec{v}_3 - \mathrm{proj}_{\vec{v}_1,\vec{v}_2}(\vec{v}_3)$, and observe:

(a) $\vec{v}_3^{\perp}$ is perpendicular to $\vec{v}_1$ and $\vec{v}_2$; and

(b) $\vec{v}_3^{\perp}$ is in $\mathrm{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$;

And then divide $\vec{v}_3^{\perp}$ by its length to obtain a unit vector $\vec{u}_3$ parallel to $\vec{v}_3^{\perp}$:

$$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_2) - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2 \qquad \text{and} \qquad \vec{u}_3 = \frac{1}{\|\vec{v}_3^{\perp}\|}\vec{v}_3^{\perp}$$

…and so on… until:

Step *k*. Define $\vec{v}_k^{\perp} = \vec{v}_k - \mathrm{proj}_{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_{k-1}}(\vec{v}_k)$, and observe:

(a) $\vec{v}_k^{\perp}$ is perpendicular to $\mathrm{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k-1}\}$; and

(b) $\vec{v}_k^{\perp}$ is in $\mathrm{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$;

And then divide $\vec{v}_k^{\perp}$ by its length to obtain a unit vector $\vec{u}_k$ parallel to $\vec{v}_k^{\perp}$:

$$\vec{v}_k^{\perp} = \vec{v}_k - (\vec{u}_1 \cdot \vec{v}_k)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_k)\vec{u}_2 - \cdots - (\vec{u}_{k-1} \cdot \vec{v}_k)\vec{u}_{k-1} \qquad \text{and} \qquad \vec{u}_k = \frac{1}{\|\vec{v}_k^{\perp}\|}\vec{v}_k^{\perp}$$

Result: The vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$ are orthonormal, and $\mathrm{span}\{\vec{u}_1, \dots, \vec{u}_j\} = \mathrm{span}\{\vec{v}_1, \dots, \vec{v}_j\}$ for all $j \leqslant k$. Therefore, if $\vec{v}_1, \dots, \vec{v}_k$ is a basis of V, then $\vec{u}_1, \dots, \vec{u}_k$ is an orthonormal basis of V.

1. Use the Gram–Schmidt process to turn the following sets of vectors into orthonormal sets spanning the same subspace.

(a) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

2. Find an orthonormal basis of the plane $x - 2y + z = 0$.

3. Define a matrix A by

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 2 & 3 & 1 & -1 \end{pmatrix}$$

(a) Find the rank of A.

(b) Find an orthonormal basis of the image of A.

(c) Find an orthonormal basis of the kernel of A.