Math 290-2 Class 2

Wednesday 9th January 2019

Ortho-more-mal

Recall that vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$ in $\mathbb{R}^n$ are orthonormal if and only if

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and that the **orthogonal projection** of a vector $\vec{x}$ onto a subspace V with orthonormal basis $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$ is given by

$$\text{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 + \cdots + (\vec{u}_k \cdot \vec{x})\vec{u}_k$$

Consequently, if $\mathfrak{B} = \vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ is an orthonormal basis of $\mathbb{R}^n$, then $\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n$.

This makes computing coordinate vectors with respect to orthonormal bases extremely easy:

$$[\vec{x}]_{\mathfrak{B}} = \begin{pmatrix} \vec{u}_1 \cdot \vec{x} \\ \vec{u}_2 \cdot \vec{x} \\ \vdots \\ \vec{u}_n \cdot \vec{x} \end{pmatrix}$$

The **orthogonal complement** of a subspace V of $\mathbb{R}^n$ is the subspace $V^\perp$ of $\mathbb{R}^n$ consisting of all vectors perpendicular to those in V:

$$V^\perp = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{v} \cdot \vec{x} = 0 \text{ for all } \vec{v} \text{ in } V\} = \ker(\text{proj}_V)$$

Note that $\dim(V) + \dim(V^\perp) = n$ by the rank–nullity theorem, since $V = \text{im}(\text{proj}_V)$.

Some geometry

Dot products, lengths and angles all neatly related by the following theorem: if $\vec{x}$ and $\vec{y}$ are any two vectors in $\mathbb{R}^n$, such that the angle between $\vec{x}$ and $\vec{y}$ is θ (where $0 \leqslant \theta \leqslant \pi$), then

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos\theta$$

Some more fun facts:

- $|\vec{x} \cdot \vec{y}| \leqslant \|\vec{x}\| \|\vec{y}\|$ — this is called the *Cauchy–Schwarz inequality*;
- $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ if and only if $\vec{x}$ and $\vec{y}$ are orthogonal.

1. (a) Verify that $\mathfrak{B} = \begin{pmatrix} 1/2 \\ 0 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -\sqrt{3}/2 \\ 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an orthonormal basis of $\mathbb{R}^3$.

(b) Find the coordinates of $\vec{a} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ with respect to $\mathfrak{B}$.

2. Let V be the plane in $\mathbb{R}^3$ spanned by $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(a) Find the orthogonal projection of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ onto V;

(b) Find the orthogonal complement of V.

3. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be vectors in $\mathbb{R}^3$ defined by

$$\vec{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

(a) Show that $\vec{c}$ is in the orthogonal complement of $\text{span}\{\vec{a}, \vec{b}\}$.

(b) Find the angle between $\vec{a}$ and $\vec{b}$.

4. For each of the following (true) statements, explain why it is true.

(a) If $|\vec{x} \cdot \vec{y}| = \|\vec{x}\| \|\vec{y}\|$, then $\vec{x}$ and $\vec{y}$ are parallel.

(b) Let ℓ be a line in $\mathbb{R}^n$ and let $\vec{v}$ and $\vec{w}$ be nonzero vectors in $\mathbb{R}^n$. If $\vec{v}$ is parallel to ℓ, and the equation $\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$ holds, then $\vec{w}$ is in $\ell^\perp$.

(c) Let V be a subspace of $\mathbb{R}^n$ and let $\vec{x}$ be a vector in $\mathbb{R}^n$. Then $\|\text{proj}_V(\vec{x})\| \leqslant \|\vec{x}\|$.