1. Find a unit vector in $\mathbb{R}^5$ that is parallel to the vector $\begin{pmatrix} 3 \\ 0 \\ -1 \\ 2 \\ 1 \end{pmatrix}$. $= \vec{v}$

$$\|\vec{v}\| = \sqrt{3^2 + 0^2 + (-1)^2 + 2^2 + 1^2} = \sqrt{15}$$

$\Rightarrow$ a unit vector parallel to $\begin{pmatrix} 3 \\ 0 \\ -1 \\ 2 \\ 1 \end{pmatrix}$ is

$$\frac{1}{\sqrt{15}} \begin{pmatrix} 3 \\ 0 \\ -1 \\ 2 \\ 1 \end{pmatrix}$$

2. Find all values of a and b for which the vectors $\begin{pmatrix} a/2 \\ b/3 \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ are orthonormal.

- <u>Orthogonal</u>: $\begin{pmatrix} a/2 \\ b/3 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{a}{2\sqrt{2}} - \frac{b}{3\sqrt{2}} = 0 \Leftrightarrow b = \frac{3a}{2}$
- <u>Unit vectors</u>: $\left\| \begin{pmatrix} a/2 \\ b/3 \end{pmatrix} \right\|^2 = \frac{a^2}{4} + \frac{b^2}{9} = \frac{a^2}{4} + \frac{9a^2}{9 \cdot 4} = \frac{a^2}{2}$

  which $= 1 \Leftrightarrow a = \pm\sqrt{2}$ (and then $b = \frac{\pm 3\sqrt{2}}{2}$)

  $\left\| \begin{matrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{matrix} \right\|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$

So:

$\begin{pmatrix} a/2 \\ b/3 \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ are orthonormal if and only if

either
- $a = \sqrt{2}$ and $b = \frac{3\sqrt{2}}{2}$

or
- $a = -\sqrt{2}$ and $b = -\frac{3\sqrt{2}}{2}$

3. (a) Find an orthonormal basis for the plane V in $\mathbb{R}^3$ given by the equation $x - y = 0$.

The vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ lie in V and are orthogonal, so let $\vec{u} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\vec{v} = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

Then $\vec{u} \perp \vec{v}$ and $\|\vec{u}\| = 1$ ($\left\|\begin{pmatrix}1\\1\\0\end{pmatrix}\right\| = \sqrt{2}$)

and $\|\vec{v}\| = 1$ (it's a standard basis vector)

$\Rightarrow$ $\vec{u}$ and $\vec{v}$ are orthonormal vectors in V

$\Rightarrow$ they're LI, so form a basis of V

(b) Find the orthogonal projection of $\vec{x} = \begin{pmatrix}2\\1\\2\end{pmatrix}$ onto the plane V from part (a).

$$\vec{u}\cdot\vec{x} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}2\\1\\2\end{pmatrix} = \frac{1}{\sqrt{2}}(2+1+0) = \frac{3}{\sqrt{2}}$$

$$\vec{v}\cdot\vec{x} = \begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\2\end{pmatrix} = 0+0+2 = 2$$

$$\Rightarrow \operatorname{proj}_V(\vec{x}) = (\vec{u}\cdot\vec{x})\vec{u} + (\vec{v}\cdot\vec{x})\vec{v}$$

$$= \frac{3}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}3/2\\3/2\\2\end{pmatrix}$$

$$\left(= \underline{\frac{1}{2}\begin{pmatrix}3\\3\\4\end{pmatrix}}\right)$$

4. For each of the following statements, determine whether it is always, sometimes or never true.

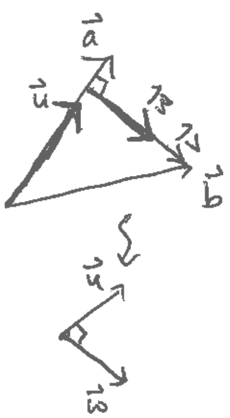
(a) Let $\vec{u}, \vec{v}, \vec{w}$ be unit vectors in $\mathbb{R}^n$. If $\vec{u}$ and $\vec{v}$ are orthonormal, and $\vec{v}$ and $\vec{w}$ are orthonormal, then $\vec{u}$ and $\vec{w}$ are orthonormal.

$\boxed{\text{Sometimes}}$

- True if $\vec{u} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\vec{v} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\vec{w} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ — they're standard basis vectors so any two of them are orthonormal.
- False if $\vec{u} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\vec{v} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\vec{w} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ — again they're standard basis vectors $\Rightarrow \vec{u}, \vec{v}$ & $\vec{v}, \vec{w}$ are orthonormal, but $\vec{u} = \vec{w} = \begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow \vec{u} \cdot \vec{w} = 1 \neq 0$

  So $\vec{u}, \vec{w}$ aren't orthonormal

(b) Let V be a plane in $\mathbb{R}^3$. Then V has an orthonormal basis.



$\boxed{\text{Always}}$ Let $\vec{a}, \vec{b}$ be a basis of the plane.

Define $\vec{u} = \frac{1}{\|\vec{a}\|}\vec{a} \Rightarrow \vec{u}$ is a unit vector in the plane.

Define $\vec{v} = \vec{b} - \text{proj}_{\vec{u}}(\vec{b}) \Rightarrow \vec{v}$ is orthogonal to $\vec{u}$ and lies in the plane ($\because$ $\vec{b}$ & $\text{proj}_{\vec{u}}(\vec{b})$ are in the plane)

Define $\vec{w} = \frac{1}{\|\vec{v}\|}\vec{v} \Rightarrow \vec{w}$ is a unit vector parallel to $\vec{v}$ ($\Rightarrow$ orthogonal to $\vec{u}$ & in the plane) $\Rightarrow$ $\boxed{\vec{u}, \vec{w} \text{ is an orthonormal basis of } V}$

(c) The vectors $\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ and $\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$ are orthonormal.

$\boxed{\text{Always}}$

- $\left\|\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right\|^2 = \left\|\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right\|^2 = \cos^2\theta + \sin^2\theta = 1$

  $\Rightarrow$ they're unit vectors
- $\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \cdot \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$

  $\Rightarrow$ they're orthonormal

So they form an orthonormal set.