Math 290-2 Class 1

Monday 7th January 2019

The dot product revisited

Recall from Math 290-1 that the dot product of vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ is given by

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

and that vectors $\vec{v}$ and $\vec{w}$ are **perpendicular** if and only if $\vec{v} \cdot \vec{w} = 0$.

The **length** of a vector $\vec{v}$ is defined to be $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$. For example,

$$\left\| \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\| = \sqrt{1^2 + (-2)^2} = \sqrt{5} \quad \text{and} \quad \left\| \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} \right\| = \sqrt{(-2)^2 + 4^2 + 1^2} = \sqrt{21}$$

A vector $\vec{u}$ is a **unit vector** if its length is 1. Some fun facts about lengths:

- $\|\vec{v}\| \geqslant 0$ for all vectors $\vec{v}$;
- If $\|\vec{v}\| = 0$, then $\vec{v} = \vec{0}$;
- If k is any scalar, then $\|k\vec{v}\| = |k|\|\vec{v}\|$;
- If $\vec{v} \neq \vec{0}$, then $\dfrac{\vec{v}}{\|\vec{v}\|}$ is a unit vector parallel to $\vec{v}$.

Orthonormal vectors

Vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$ in $\mathbb{R}^n$ are said to be **orthonormal** if they are unit vectors and if they are pairwise orthogonal. Or, more succinctly, if:

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Some fun facts about orthonormal vectors:

- The standard basis vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ are orthonormal;
- If $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$ are orthonormal, then they are linearly independent;
- Any collection of n orthonormal vectors in $\mathbb{R}^n$ forms a basis of $\mathbb{R}^n$.

A basis $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$ of a subspace V of $\mathbb{R}^n$ is called an **orthonormal basis** of V.

Every vector $\vec{x}$ can be written uniquely as $\vec{x}^{\parallel} + \vec{x}^{\perp}$, where $\vec{x}^{\parallel}$ is in V and $\vec{x}^{\perp}$ is perpendicular to V. The vector $\vec{x}^{\parallel}$ is the **orthogonal projection** $\vec{x}$ onto V, and is given by

$$\vec{x}^{\parallel} = \text{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 + \cdots + (\vec{u}_k \cdot \vec{x})\vec{u}_k$$

1. Find a unit vector in $\mathbb{R}^5$ that is parallel to the vector $\begin{pmatrix} 3 \\ 0 \\ -1 \\ 2 \\ 1 \end{pmatrix}$.

2. Find all values of a and b for which the vectors $\begin{pmatrix} a/2 \\ b/3 \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ are orthonormal.

3. (a) Find an orthonormal basis for the plane V in $\mathbb{R}^3$ given by the equation $x - y = 0$.

(b) Find the orthogonal projection of $\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ onto the plane V from part (a).

4. For each of the following statements, determine whether it is always, sometimes or never true.

(a) Let $\vec{u}, \vec{v}, \vec{w}$ be unit vectors in $\mathbb{R}^n$. If $\vec{u}$ and $\vec{v}$ are orthonormal, and $\vec{v}$ and $\vec{w}$ are orthonormal, then $\vec{u}$ and $\vec{w}$ are orthonormal.

(b) Let V be a plane in $\mathbb{R}^3$. Then V has an orthonormal basis.

(c) The vectors $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ are orthonormal.