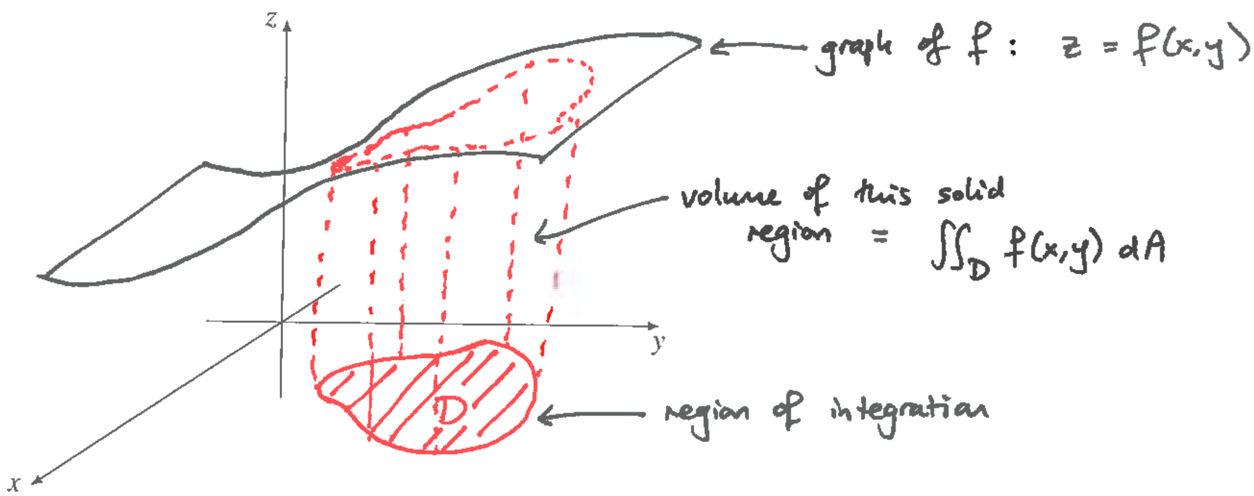
Math 290-3 Class 1

Monday 1st April 2019

Double integrals

A bounded integral $\int_a^b f(x)\,dx$ tells us the area under the curve $y = f(x)$ above the interval $[a,b] = \{x : a \leqslant x \leqslant b\}$. Intuitively, the integral adds up the heights of the points $(x, f(x))$ for $a \leqslant x \leqslant b$.

Double integrals are the generalisation of (bounded) integrals to functions of two variables: the double integral $\iint_D f(x,y)\,dA$ tells us the *volume* under the *surface* $z = f(x,y)$ above the region D of the (x,y)-plane.



When D is the square region $[a,b] \times [c,d] = \{(x,y) : a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$ and f is sufficiently well-behaved* on D, there are two ways that we can compute $\iint_D f(x,y)\,dA$:

- Find the areas under the curves $z = f(x,y)$ for fixed $a \leqslant x \leqslant b$ (by integrating with respect to y, holding x constant); then 'add up' these areas by integrating with respect to x:

$$\boxed{\iint_{[a,b]\times[c,d]} f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx}$$

- Find the areas under the curves $z = f(x,y)$ for fixed $c \leqslant y \leqslant d$ (by integrating with respect to x, holding y constant); then 'add up' these areas by integrating with respect to y:

$$\boxed{\iint_{[a,b]\times[c,d]} f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy}$$

Note that, in particular, the two iterated integrals are equal—this fact is called **Fubini's theorem.**

[*Every function we will encounter is 'sufficiently well-behaved' for the purposes of applying Fubini's theorem.]

1. Compute $\iint_{[1,2]\times[-1,1]} xe^{xy}\, dA\ldots$

(a) ...by first integrating with respect to y and then with respect to x.

$$\int_1^2 \int_{-1}^1 x e^{xy}\, dy\, dx = \int_1^2 \left[e^{xy}\right]_{y=-1}^{y=1} dx$$
$$= \int_1^2 (e^x - e^{-x})\, dx$$
$$= \left[e^x + e^{-x}\right]_1^2$$
$$= e^2 + e^{-2} - e - e^{-1}$$

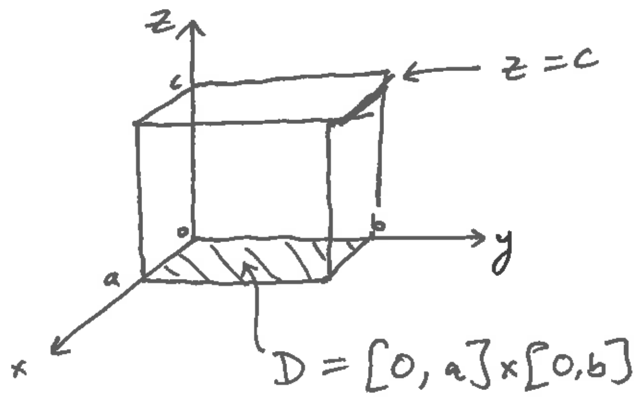
(b) ...by first integrating with respect to x and then with respect to y.

$u = x \Rightarrow du = dx$
$dv = e^{xy} \Rightarrow v = \frac{e^{xy}}{y}$

$$\int_{-1}^1 \int_1^2 x e^{xy}\, dx\, dy = \int_{-1}^1 \left(\left[\frac{xe^{xy}}{y}\right]_1^2 - \int_1^2 \frac{e^{xy}}{y}\, dx \right) dy$$
$$= \int_{-1}^1 \left(\frac{2e^{2y}}{y} - \frac{e^y}{y} - \left[\frac{e^{xy}}{y^2}\right]_1^2 \right) dy$$
$$= \int_{-1}^1 \left(\frac{2e^{2y}}{y} - \frac{e^y}{y} - \frac{e^{2y}}{y^2} - \frac{e^y}{y^2} \right) dy$$
$$= \ldots \text{ uh-oh!}$$

There is no closed form expression for $\int \frac{e^y}{y}\, dy$ for example.

2. Use double integration to show that the volume of a cube of width a, length b and height c is equal to abc.



$$\text{Volume} = \iint_D c \, dA$$

$$= \int_0^a \int_0^b c \, dy \, dx$$

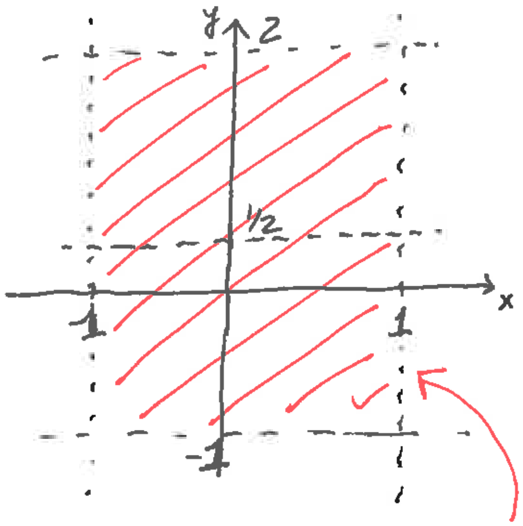
$$= \int_0^a [cy]_0^b \, dx$$

$$= \int_0^a bc \, dx$$

$$= [bcx]_0^a$$

$$= \underline{\underline{abc}}$$

3. Find the volume of the solid bounded by the (x, y)-plane, the plane $x = 1$, the plane $x = -1$, the plane $z = 1 + y$ and the plane $z = 2 - y$.

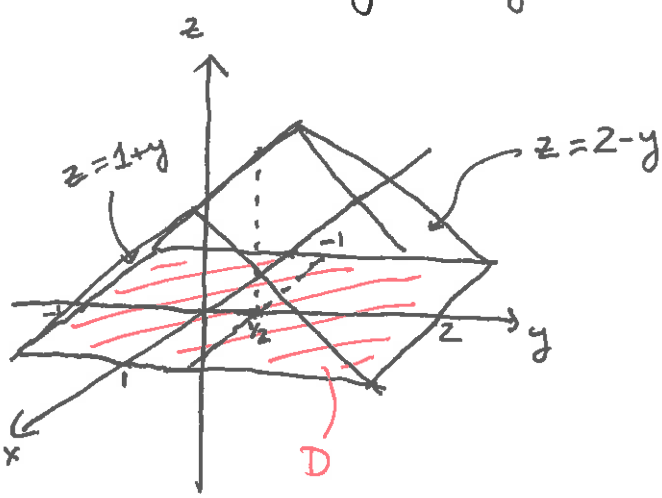


- $z = 1 + y$ intersects the (x,y)-plane at $0 = 1 + y \Rightarrow y = -1$
- $z = 2 - y$ intersects the (x,y)-plane at $2 - y = 0 \Rightarrow y = 2$

$D = [-1,1] \times [-1,2]$

The planes $z = 1 + y$ & $z = 2 - y$ intersect when

$$1 + y = 2 - y \Rightarrow 2y = 1 \Rightarrow y = \frac{1}{2}$$



So volume $= \iint_D z \, dA$

$$= \iint_{[-1,1]\times[-1,\frac{1}{2}]} 1+y \, dA + \iint_{[-1,1]\times[\frac{1}{2},2]} 2-y \, dA$$

$$= \int_{-1}^{1}\int_{-1}^{\frac{1}{2}} (1+y)\,dy\,dx + \int_{-1}^{1}\int_{\frac{1}{2}}^{2} (2-y)\,dy\,dx$$

$$= \int_{-1}^{1} \left[\frac{(1+y)^2}{2}\right]_{-1}^{\frac{1}{2}} dx + \int_{-1}^{1} \left[-\frac{(2-y)^2}{2}\right]_{\frac{1}{2}}^{2} dx$$

$$= \int_{-1}^{1} \frac{9}{8}\,dx + \int_{-1}^{1} \frac{9}{8}\,dx$$

$$= 2 \cdot \frac{9}{8} + 2 \cdot \frac{9}{8}$$

$$= \frac{9}{2}$$