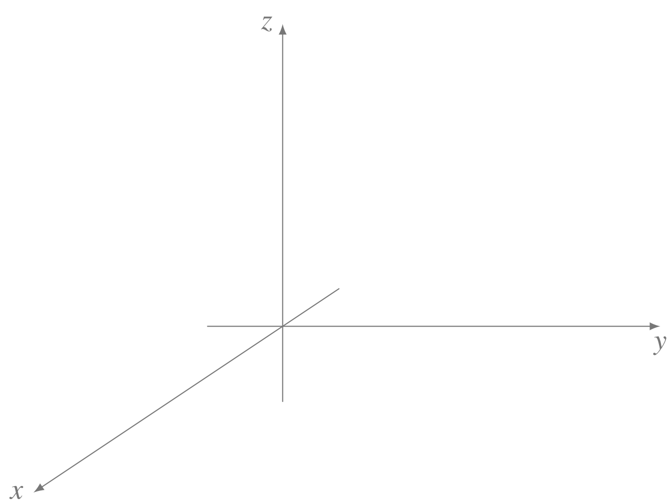
Math 290-3 Class 1

Monday 1st April 2019

Double integrals

A bounded integral $\int_a^b f(x)\,dx$ tells us the area under the curve $y = f(x)$ above the interval $[a,b] = \{x : a \leqslant x \leqslant b\}$. Intuitively, the integral adds up the heights of the points $(x, f(x))$ for $a \leqslant x \leqslant b$.

Double integrals are the generalisation of (bounded) integrals to functions of two variables: the double integral $\iint_D f(x,y)\,dA$ tells us the *volume* under the *surface* $z = f(x,y)$ above the region D of the (x,y)-plane.



When D is the square region $[a,b] \times [c,d] = \{(x,y) : a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$ and f is sufficiently well-behaved* on D, there are two ways that we can compute $\iint_D f(x,y)\,dA$:

- Find the areas under the curves $z = f(x,y)$ for fixed $a \leqslant x \leqslant b$ (by integrating with respect to y, holding x constant); then 'add up' these areas by integrating with respect to x:

$$\boxed{\iint_{[a,b]\times[c,d]} f(x,y)\,dA \;=\; \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx}$$

- Find the areas under the curves $z = f(x,y)$ for fixed $c \leqslant y \leqslant d$ (by integrating with respect to x, holding y constant); then 'add up' these areas by integrating with respect to y:

$$\boxed{\iint_{[a,b]\times[c,d]} f(x,y)\,dA \;=\; \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy}$$

Note that, in particular, the two iterated integrals are equal—this fact is called **Fubini's theorem**.

[*Every function we will encounter is 'sufficiently well-behaved' for the purposes of applying Fubini's theorem.]

1. Compute $\iint_{[1,2]\times[-1,1]} xe^{xy}\, dA\ldots$

(a) …by first integrating with respect to y and then with respect to x.

(b) …by first integrating with respect to x and then with respect to y.

2. Use double integration to show that the volume of a cube of width a, length b and height c is equal to abc.

3. Find the volume of the solid bounded by the (x, y)-plane, the plane $x = 1$, the plane $x = -1$, the plane $z = 1 + y$ and the plane $z = 2 - y$.