1. Find the eigenvalues of the following matrix, and find the algebraic and geometric multiplicity of each eigenvalue.

$$\underbrace{\begin{pmatrix} 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{:= A}$$

$$f_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 & & & \\ & 2-\lambda & & & \\ & & -1-\lambda & & \\ & & & -1-\lambda & \\ & & & & -\lambda \end{vmatrix} = (2-\lambda)^2(-1-\lambda)^2(-\lambda)$$

So the eigenvalues are 2 (alg.mu. $= \underline{\underline{2}}$), -1 (alg.mu. $= \underline{\underline{2}}$), 0 (alg. mu. $= \underline{\underline{1}}$)

$\underline{\underline{\lambda = 2}}$ $\quad A - 2I = \begin{pmatrix} 0 & 2 & & & \\ & 0 & & & \\ & & -3 & & \\ & & & -3 & \\ & & & & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\uparrow$ free

$x_1 = t$ free, $\quad x_2 = x_3 = x_4 = x_5 = 0 \quad \Rightarrow \quad \vec{x} = \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \underset{\substack{\parallel \\ E_2}}{\ker(A - 2I)} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\} \Rightarrow$ geo.mu. of 2 is $\underline{\underline{1}}$

$\underline{\underline{\lambda = -1}}$ $\quad A + I = \begin{pmatrix} 3 & 2 & & & \\ & 3 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\uparrow \ \uparrow$ free

$x_3 = s$, $x_4 = t$ free, $\quad x_1 = x_2 = x_5 = 0 \Rightarrow \vec{x} = \begin{pmatrix} 0 \\ 0 \\ s \\ t \\ 0 \end{pmatrix}$

$\Rightarrow E_{-1} = \ker(A + I) = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$

$\Rightarrow$ geo. mu. of -1 is $\underline{\underline{2}}$

$\underline{\underline{\lambda = 0}}$ $\quad A + 0I = A \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E_0 = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\} \Rightarrow$ geo. mu. of 0 is $\underline{\underline{1}}$.

2. For each of the following, determine whether it is always, sometimes or never true.

(a) Let A be a 4×4 matrix with eigenvalues 1, 2 and 3. Exactly one of 1, 2 or 3 has algebraic multiplicity $\geqslant 2$.

Always True! $f_A(\lambda) = \pm(\lambda-1)(\lambda-2)(\lambda-3)p(\lambda)$ for some $p(\lambda)$

$\deg(f_A) = 4 \Rightarrow \deg(p) = 1 \Rightarrow p(\lambda) = \lambda - a$ for some a

The only eigenvalues of A are $1, 2, 3 \Rightarrow a = 1, 2$ or 3

& the eigenvalue a has alg. mu. $= 2$, all other alg. mu. $= 1$.

(b) Let A be an $n \times n$ matrix with n distinct eigenvalues. Then the geometric multiplicities of the eigenvalues of A add up to n.

Always True! E_{λ_i} has dimension $\geqslant 1$ for each λ_i, and since the λ_i are distinct, the corresponding eigenvectors $\vec{v_i}$ are LI $\Rightarrow$ they form a basis of $\mathbb{R}^n$.

$\Rightarrow E_{\lambda_i} = \operatorname{span}\{\vec{v_i}\}$ for each i $\Rightarrow$ each e.vec. has geo. mu. $= 1$

$\Rightarrow$ the geo. mu. add up to n.

(c) Let A and B be $n \times n$ matrices such that $f_A(\lambda) = f_B(\lambda)$. Then A and B are similar.

Sometimes true!

- If $A = B = I_2$ then $f_A(\lambda) = f_B(\lambda) = (1-\lambda)^2$ & A and B are trivially similar.
- If $A = I_2$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then $f_A(\lambda) = f_B(\lambda) = (1-\lambda)^2$ but A and B are not similar by Class 17 Q2(b).

(d) Let A be a 3×3 matrix and suppose that there are nonzero vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ such that $A\vec{u} = -\vec{u}$, $A\vec{v} = 2\vec{v}$ and $A\vec{w} = \vec{0}$. There is some nonzero vector $\vec{x}$ such that $A\vec{x} = 4\vec{x}$.

Never true! $f_A(\lambda)$ is a cubic polynomial

& $f_A(\lambda) = 0$ when $\lambda = -1$, 2 and 0, since $\vec{u}, \vec{v}, \vec{w}$ are eigenvectors of A

$\Rightarrow f_A(\lambda) = (-1-\lambda)(2-\lambda)(-\lambda)$

$\Rightarrow f_A(4) = (-5)(-2)(-4) = -40 \neq 0$

$\Rightarrow$ A has no eigenvector w. eigenvalue 4.

3. [Bretscher §7.3 Q21] Find a 2×2 matrix A such that $E_1 = \text{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ and $E_2 = \text{span}\left\{\begin{pmatrix}2\\3\end{pmatrix}\right\}$.
How many such matrices are there?

Let $\vec{v}_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}2\\3\end{pmatrix}$. Then $\mathcal{B} = \vec{v}_1, \vec{v}_2$ is a basis of $\mathbb{R}^2$, and since $A\vec{v}_1 = \vec{v}_1$ and $A\vec{v}_2 = 2\vec{v}_2$ the $\mathcal{B}$-matrix B of A is given by

$$B = \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$$

But then

$$A = \underbrace{S}_{=\text{ transition mx of } \mathcal{B}} B S^{-1} = \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}^{-1}$$

$$= \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\cdot\left(-\begin{pmatrix}3 & -2\\-2 & 1\end{pmatrix}\right)$$

$$= -\begin{pmatrix}1 & 4\\2 & 6\end{pmatrix}\begin{pmatrix}3 & -2\\-2 & 1\end{pmatrix}$$

$$= -\begin{pmatrix}-5 & 2\\-6 & 2\end{pmatrix}$$

$$= \begin{pmatrix}5 & -2\\6 & -2\end{pmatrix}$$

Check: $\begin{pmatrix}5 & -2\\6 & -2\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}5-4\\6-4\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

$\begin{pmatrix}5 & -2\\6 & -2\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}10-6\\12-6\end{pmatrix} = \begin{pmatrix}4\\6\end{pmatrix} = 2\begin{pmatrix}2\\3\end{pmatrix}$

So A is as required. Moreover A is the only such matrix — the vectors $\vec{v}_1, \vec{v}_2$ determined the $\mathcal{B}$-matrix of A, which in turn determined A.

[Warning: a matrix might not have a basis of eigenvectors. In this case we were lucky!]

4. A *Jordan block* is an $n \times n$ matrix with a single scalar λ in its diagonal entries, 1 in each entry immediately the right of a diagonal entry, and 0 in all other entries:

$$J_{\lambda,n} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

(a) Find the algebraic and geometric multiplicities of the eigenvalue λ in $J_{\lambda,n}$.

$\det(J - \lambda_0 I_n) = (\lambda - \lambda_0)^n \Rightarrow$ alg. mult. $\underline{\underline{= n}}$

$J - \lambda_0 I_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$ which is in rref!

$\Rightarrow \operatorname{rank}(J - \lambda I) = n - 1$

$\Rightarrow$ geo. mult. of λ $= \dim(E_\lambda) = n - (n-1) \underline{\underline{= 1}}$.

(b) Let λ be a scalar, and let A be a 2×2 matrix whose only eigenvalue is λ. Show that A is similar to either λI_2 or $J_{\lambda,2}$.

- If $\dim(E_\lambda) = 2$ then $E_\lambda = \mathbb{R}^2 \Rightarrow A\vec{e}_1 = \lambda\vec{e}_1$ & $A\vec{e}_2 = \lambda\vec{e}_2$
  $\Rightarrow A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \underline{\underline{\lambda I_2}}$.

- Suppose $\dim(E_\lambda) = 1$. First note $A - \lambda I$ is similar to $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ for some $* \neq 0$, since if $\vec{a}$ is in E_λ & $\vec{b}$ is perp. to $\vec{a}$ then the $(\vec{a}, \vec{b})$-mx of A is $\begin{pmatrix} \lambda & * \\ 0 & \lambda \end{pmatrix}$ for some $* \neq 0$.
  since $A\vec{a} = \lambda\vec{a}$ since λ has alg. mult. $= 2$ & geo. mult. < 2.

$\Rightarrow (A - \lambda I)^2$ is similar ($\Rightarrow$ equal) to $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Let $\vec{w}$ be s.t. $(A - \lambda I)\vec{w} \neq \vec{0}$ & let $\vec{v} = (A - \lambda I)\vec{w}$.

Then the $(\vec{v}, \vec{w})$-mx of $A - \lambda I$ is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

since $(A - \lambda I)\vec{v} = (A - \lambda I)^2\vec{w} = \vec{0}$ since $(A - \lambda I)\vec{w} = \vec{v}$

$\Rightarrow$ the $(\vec{v}, \vec{w})$-mx of A is $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \underline{\underline{J_{\lambda,2}}}$.