Math 290-1 Class 23

Monday 19th November 2018

More on eigenvectors and eigenvalues

Quick facts from last time:

- The **characteristic polynomial** of an $n \times n$ matrix A is $f_A(\lambda) = \det(A - \lambda I_n)$.
- The **eigenvalues** of A are the roots of $f_A(\lambda)$ (i.e. the solutions to $f_A(\lambda) = 0$).
- The **eigenvectors** of A are the nonzero vectors in $\ker(A - \lambda I_n)$.

Some consequences:

- If A is similar to B, then $f_A(\lambda) = f_B(\lambda)$;
- If $f_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$, then

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n \quad \text{and} \quad \operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Geometric multiplicity

The *geometric multiplicity* of an eigenvalue λ tells us how many linearly independent eigenvectors with eigenvalue λ there are.

- The **λ-eigenspace** of T is the subspace $E_\lambda = \ker(A - \lambda I_n)$ of $\mathbb{R}^n$.
  The vectors in the λ-eigenspace are exactly ($\vec{0}$ and) the eigenvectors of T with eigenvalue λ.
- The **geometric multiplicity** of the eigenvalue λ is $\dim(E_\lambda)$. By rank–nullity, we have:

$$\begin{matrix}\text{geometric multiplicity} \\ \text{of } \lambda\end{matrix} \quad = \quad \dim(E_\lambda) \quad = \quad n - \operatorname{rank}(A - \lambda I_n)$$

Algebraic multiplicity

The *algebraic multiplicity* of an eigenvalue is its multiplicity as a root of $f_A(\lambda)$. For example, if

$$f_A(\lambda) \ = \ \lambda^6 - 12\lambda^5 + 45\lambda^4 - 30\lambda^3 - 120\lambda^2 + 96\lambda + 128 \ = \ (\lambda+1)^2(\lambda-2)(\lambda-4)^3$$

then the eigenvalues of A are -1, 2 and 4, and:

- The algebraic multiplicity of the eigenvalue -1 is 2;
- The algebraic multiplicity of the eigenvalue 2 is 1; and
- The algebraic multiplicity of the eigenvalue 4 is 3.

The algebraic multiplicity of an eigenvalue is greater than or equal to its geometric multiplicity.

1. Find the eigenvalues of the following matrix, and find the algebraic and geometric multiplicity of each eigenvalue.

$$\begin{pmatrix} 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2. For each of the following, determine whether it is always, sometimes or never true.

(a) Let A be a 4×4 matrix with eigenvalues 1, 2 and 3. Exactly one of 1, 2 or 3 has algebraic multiplicity $\geqslant 2$.

(b) Let A be an $n \times n$ matrix with n distinct eigenvalues. Then the geometric multiplicities of the eigenvalues of A add up to n.

(c) Let A and B be $n \times n$ matrices such that $f_A(\lambda) = f_B(\lambda)$. Then A and B are similar.

(d) Let A be a 3×3 matrix and suppose that there are nonzero vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ such that $A\vec{u} = -\vec{u}$, $A\vec{v} = 2\vec{v}$ and $A\vec{w} = \vec{0}$. There is some nonzero vector $\vec{x}$ such that $A\vec{x} = 4\vec{x}$.

3. [Bretscher §7.3 Q21] Find a 2×2 matrix A such that $E_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ and $E_2 = \text{span}\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$. How many such matrices are there?

4. A *Jordan block* is an $n \times n$ matrix with a single scalar λ in its diagonal entries, 1 in each entry immediately the right of a diagonal entry, and 0 in all other entries:

$$J_{\lambda,n} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

(a) Find the algebraic and geometric multiplicities of the eigenvalue λ in $J_{\lambda,n}$.

(b) Let λ be a scalar, and let A be a 2×2 matrix whose only eigenvalue is λ. Show that A is similar to either λI_2 or $J_{\lambda,2}$.