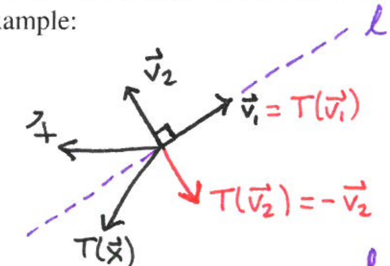
Math 290-1 Class 22

Friday 16th November 2018

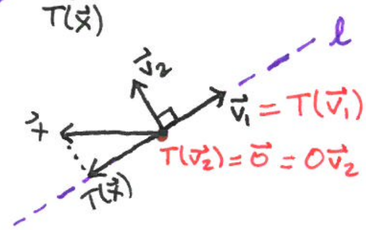
Eigenvectors and eigenvalues

An **eigenvector** of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is a nonzero(!) vector $\vec{v}$ such that $T(\vec{v}) = \lambda \vec{v}$ for some scalar λ, called the **eigenvalue** of $\vec{v}$. For example:

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by reflection through a line ℓ, then any nonzero vector $\vec{v}$ parallel to ℓ is an eigenvector of T with eigenvalue 1, and any nonzero vector $\vec{w}$ perpendicular to ℓ is an eigenvector of T with eigenvalue -1.



If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by orthogonal projection onto a line ℓ, then any nonzero vector $\vec{v}$ parallel to ℓ is an eigenvector of T with eigenvalue 1, and any nonzero vector $\vec{w}$ perpendicular to ℓ is an eigenvector of T with eigenvalue 0.



Finding eigenvectors and eigenvalues

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with matrix A, we have

$$T(\vec{v}) = \lambda \vec{v} \quad \Leftrightarrow \quad (A - \lambda I_n)\vec{v} = \vec{0} \quad \Leftrightarrow \quad \vec{v} \text{ is in } \ker(A - \lambda I_n)$$

for all vectors $\vec{v}$ in $\mathbb{R}^n$ and all scalars λ. This means:

- The eigenvalues of T are the solutions λ to the equation $\det(A - \lambda I_n) = 0$.
- The eigenvectors of T with eigenvalue λ are the vectors in the kernel of $A - \lambda I_n$.

The function $f_A(\lambda) = \det(A - \lambda I_n)$ is called the **characteristic polynomial** of A. Fun facts:

- $f_A(\lambda) = 0$ if and only if λ is an eigenvalue of A.
- $f_A(0) = \det(A)$, so the constant term of f_A is the determinant of A.
- If $f_A(\lambda)$ can be fully factorised:
$$f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$
then $\det(A) = f(0) = \lambda_1 \lambda_2 \cdots \lambda_n$ is the product of the eigenvalues of A.
- The **trace** of an $n \times n$ matrix is the sum of its diagonal entries, and the coefficient of λ^{n-1} in $f_A(\lambda)$ is $(-1)^{n-1}\mathrm{tr}(A)$. For example, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then
$$f_A(\lambda) \quad = \quad \lambda^2 - (a+d)\lambda + (ad - bc) \quad = \quad \lambda^2 - (\mathrm{tr}\, A)\lambda + (\det A)$$

1. Find the eigenvalues and eigenvectors of the linear transformation $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & -1 & 8 \\ 0 & 0 & 2 \end{pmatrix}$$

<u>Eigenvalues</u>

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -2 \\ 1 & -1-\lambda & 8 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

Expand along row 3

$$= (2-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 1 & -1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(1-\lambda)(-1-\lambda)$$

So the eigenvalues are 2, 1 and -1.

- $\underline{\underline{\lambda = 2}}$ $\quad A - 2I = \begin{pmatrix} -1 & 0 & -2 \\ 1 & -3 & 8 \\ 0 & 0 & 0 \end{pmatrix}$

$$\longrightarrow \begin{pmatrix} -1 & 0 & -2 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{pmatrix} (\mathrm{II}) + (\mathrm{I})$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} (-1) \times (\mathrm{I}) \\ (-1/3) \times (\mathrm{II}) \end{matrix} \qquad \begin{matrix} z = t \\ x = -2t \\ y = 2t \end{matrix}$$

$\ker(A - 2I) = \text{span}\left\{ \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right\} \Rightarrow \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector w/ eigenvalue 2

- $\underline{\underline{\lambda = 1}}$ $\quad A - I = \begin{pmatrix} 0 & 0 & -2 \\ 1 & -2 & 8 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} y = t \\ x = 2t \\ z = 0 \end{matrix}$

$\ker(A - I) = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\} \Rightarrow \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector w/ eigenvalue 1

- $\underline{\underline{\lambda = -1}}$ $\quad A + I = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 0 & 8 \\ 0 & 0 & 3 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} y = t \\ x = 0 \\ z = 0 \end{matrix}$

$\ker(A + I) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \Rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector w/ eigenvalue -1

2. Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 12 & -10 \\ 15 & -13 \end{pmatrix}$.

$\uparrow := A$

$$\mathrm{tr}(A) = 12 - 13 = -1$$

$$\det(A) = -12 \cdot 13 + 10 \cdot 15 = -156 + 150 = -6$$

$$\Rightarrow f_A(\lambda) = \lambda^2 - (-1)\lambda + (-6) = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

$\Rightarrow$ Eigenvalues are -3 and 2.

$\underline{\underline{\lambda = -3}}$ $\quad A + 3I = \begin{pmatrix} 15 & -10 \\ 15 & -10 \end{pmatrix}$

$\Rightarrow \ker(A+3I) = \mathrm{span}\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

$\Rightarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector w/ eigenvalue -3

$\underline{\underline{\lambda = 2}}$ $\quad A - 2I = \begin{pmatrix} 10 & -10 \\ 13 & -13 \end{pmatrix}$

$\Rightarrow \ker(A-2I) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector w/ eigenvalue 2.

3. Let $T : \mathbb{R}^6 \to \mathbb{R}^6$ be defined by $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{pmatrix} 2 & 3 & -1 & 4 & 2 & 6 \\ 8 & 0 & 0 & 8 & 0 & 0 \\ 10 & -4 & 5 & 5 & 1 & -1 \\ -1 & 4 & 4 & 1 & 4 & 4 \\ 20 & 5 & -3 & -3 & -3 & 0 \\ 2 & 2 & 3 & 3 & 3 & 3 \end{pmatrix}$$

By considering the sums of the numbers in each row, find an eigenvector of T.

The sums of the rows of A all equal 16

$$\Rightarrow A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 16 \\ 16 \\ 16 \\ 16 \\ 16 \end{pmatrix} = 16 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of A w/ eigenvalue 16.

4. For each of the following, determine whether it is true or false.

(a) It is possible for 0 to be an eigenvalue of an invertible linear transformation.

False! 0 is an eigenvalue of a mx A $\Leftrightarrow f_A(0) = 0$

$\Leftrightarrow \det(A) = \det(A - 0I) = 0$

$\Leftrightarrow$ A is not invertible.

(b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation given by rotation about the origin by an angle $0 < \theta < \pi$, then T has no real eigenvalues.

False! $\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta$

$= \cos^2\theta - (2\cos\theta)\lambda + \lambda^2 + \sin^2\theta$

$= \lambda^2 - 2\cos\theta\,\lambda + 1$

This has a real root $\Leftrightarrow (-2\cos\theta)^2 - 4 \geq 0 \Leftrightarrow \cos^2\theta \geq 1$

$\Leftrightarrow \cos\theta = \pm 1$ — but $|\cos\theta| < 1$ when $0 < \theta < \pi$.

since $-1 \leq \cos\theta \leq 1$ for all θ $\Rightarrow$ no real roots.

(c) If A and B are similar matrices, then A and B have the same eigenvalues.

True! If $B = S^{-1}AS$ for some invertible S, then

$f_B(\lambda) = \det(B - \lambda I) = \det(S^{-1}AS - \lambda S^{-1}IS)$ matrix algebra

$= \det(S^{-1}(A - \lambda I)S)$ similar matrices have same determinant

$= \det(A - \lambda I)$

$= f_A(\lambda) \Rightarrow f_A, f_B$ have same roots $\equiv$ eigenvalues.

(d) If A and B are similar matrices, then A and B have the same eigenvectors.

False (in general)!

Let T be reflection through $y = x$.

- Standard mx of $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} := A$.

  Eigenvalues are 1 (eigenvector $= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$), -1 (e.vec. $= \begin{pmatrix} -1 \\ 1 \end{pmatrix}$)

- Mx of T wrt the basis $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} := B$ $\left[= S^{-1}AS \text{ where } S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right]$

  Eigenvalues are 1 (eigenvector $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$), -1 (e.vec. $= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$)

But $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not in span$\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ or span$\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$

$\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not an eigenvector of A