Math 290-1 Class 22

Friday 16th November 2018

Eigenvectors and eigenvalues

An **eigenvector** of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is a nonzero(!) vector $\vec{v}$ such that $T(\vec{v}) = \lambda\vec{v}$ for some scalar λ, called the **eigenvalue** of $\vec{v}$. For example:

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by reflection through a line ℓ, then any nonzero vector $\vec{v}$ parallel to ℓ is an eigenvector of T with eigenvalue 1, and any nonzero vector $\vec{w}$ perpendicular to ℓ is an eigenvector of T with eigenvalue -1.

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by orthogonal projection onto a line ℓ, then any nonzero vector $\vec{v}$ parallel to ℓ is an eigenvector of T with eigenvalue 1, and any nonzero vector $\vec{w}$ perpendicular to ℓ is an eigenvector of T with eigenvalue 0.

Finding eigenvectors and eigenvalues

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with matrix A, we have

$$T(\vec{v}) = \lambda\vec{v} \quad \Leftrightarrow \quad (A - \lambda I_n)\vec{v} = \vec{0} \quad \Leftrightarrow \quad \vec{v} \text{ is in } \ker(A - \lambda I_n)$$

for all vectors $\vec{v}$ in $\mathbb{R}^n$ and all scalars λ. This means:

- The eigenvalues of T are the solutions λ to the equation $\det(A - \lambda I_n) = 0$.
- The eigenvectors of T with eigenvalue λ are the vectors in the kernel of $A - \lambda I_n$.

The function $f_A(\lambda) = \det(A - \lambda I_n)$ is called the **characteristic polynomial** of A. Fun facts:

- $f_A(\lambda) = 0$ if and only if λ is an eigenvalue of A.
- $f_A(0) = \det(A)$, so the constant term of f_A is the determinant of A.
- If $f_A(\lambda)$ can be fully factorised:

  $$f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

  then $\det(A) = f(0) = \lambda_1\lambda_2\cdots\lambda_n$ is the product of the eigenvalues of A.
- The **trace** of an $n \times n$ matrix is the sum of its diagonal entries, and the coefficient of λ^{n-1} in $f_A(\lambda)$ is $(-1)^{n-1}\text{tr}(A)$. For example, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

  $$f_A(\lambda) \quad = \quad \lambda^2 - (a+d)\lambda + (ad - bc) \quad = \quad \lambda^2 - (\text{tr}\, A)\lambda + (\det A)$$

1. Find the eigenvalues and eigenvectors of the linear transformation $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & -1 & 8 \\ 0 & 0 & 2 \end{pmatrix}$$

2. Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 12 & -10 \\ 15 & -13 \end{pmatrix}$.

3. Let $T : \mathbb{R}^6 \to \mathbb{R}^6$ be defined by $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{pmatrix} 2 & 3 & -1 & 4 & 2 & 6 \\ 8 & 0 & 0 & 8 & 0 & 0 \\ 10 & -4 & 5 & 5 & 1 & -1 \\ -1 & 4 & 4 & 1 & 4 & 4 \\ 20 & 5 & -3 & -3 & -3 & 0 \\ 2 & 2 & 3 & 3 & 3 & 3 \end{pmatrix}$$

By considering the sums of the numbers in each row, find an eigenvector of T.

4. For each of the following, determine whether it is true or false.

(a) It is possible for 0 to be an eigenvalue of an invertible linear transformation.

(b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation given by rotation about the origin by an angle $0 < \theta < \pi$, then T has no real eigenvalues.

(c) If A and B are similar matrices, then A and B have the same eigenvalues.

(d) If A and B are similar matrices, then A and B have the same eigenvectors.