1. Use elementary row and column operations to find the determinant of the following matrix.

$$\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{vmatrix}$$

add row 1 to rows 2, 3, 4

$$= \begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & -2 & -2 & 0 \end{vmatrix}$$

swap row 2 and row 3

$$= -\begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \end{vmatrix}$$

subtract (row 2 + row 3) from row 4

$$= -\begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

the matrix is upper-triangular

$$= -\left(1 \times (-2) \times (-2) \times 4\right)$$

$$= \underline{\underline{-16}}$$

2. Use elementary row and column operations to find determinant of the following matrix.

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 1 & 6 & 4 & 8 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & 4 & 12 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 3 & 2 & 4 \\ 1 & 6 & 4 & 8 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & 4 & 12 \end{vmatrix}$$

pull out a factor of:
3 from column 2
2 from column 3
4 from column 4

$$= \underbrace{3 \times 2 \times 4}_{=24} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 3 \end{vmatrix}$$

subtract row 1 from rows 2 and 3
& $2 \times$(row 1) from row 4

$$= 24 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

matrix is upper-triangular

$$= 24 \times 1 \times 1 \times (-1) \times 1$$

$$= \underline{\underline{-24}}$$

3. [Bretscher §6.2 Q15] Given that A is a 4×4 matrix with determinant 8, find the determinant of the matrix given by

$$\begin{pmatrix} \cdots & \vec{v}_1 & \cdots \\ \cdots & \vec{v}_1 + \vec{v}_2 & \cdots \\ \cdots & \vec{v}_1 + \vec{v}_2 + \vec{v}_3 & \cdots \\ \cdots & \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 & \cdots \end{pmatrix}$$

where $\vec{v}_i$ is the i^{th} row of A.

$$\det \begin{pmatrix} \cdots & \vec{v}_1 & \cdots \\ \cdots & \vec{v}_1 + \vec{v}_2 & \cdots \\ \cdots & \vec{v}_1 + \vec{v}_2 + \vec{v}_3 & \cdots \\ \cdots & \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 & \cdots \end{pmatrix}$$

subtract row 1 from row 2
row 2 from row 3
& row 3 from row 4

$$= \det \begin{pmatrix} \cdots & \vec{v}_1 & \cdots \\ \cdots & \vec{v}_2 & \cdots \\ \cdots & \vec{v}_3 & \cdots \\ \cdots & \vec{v}_4 & \cdots \end{pmatrix}$$

$$= \det(A) = \underline{\underline{8.}}$$

4. [Bretscher §6.2 Q48] Let $T : \mathbb{R}^n \to \mathbb{R}$ be the linear transformation defined by

$$T(\vec{x}) = \det \begin{pmatrix} \vdots & \vdots & & \vdots & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \vec{x} \\ \vdots & \vdots & & \vdots & \vdots \end{pmatrix}$$

where $\vec{v}_1, \ldots, \vec{v}_{n-1}$ are linearly independent column vectors in $\mathbb{R}^n$. Describe the kernel of T.

$\vec{x} \in \ker(T) \iff T(\vec{x}) = \vec{0}$

"is in"

$\iff$ the matrix is non-invertible

$\iff$ the columns of the mx are linearly dependent

Since we know that the vectors $\vec{v}_1, \ldots, \vec{v}_{n-1}$ are L.I.

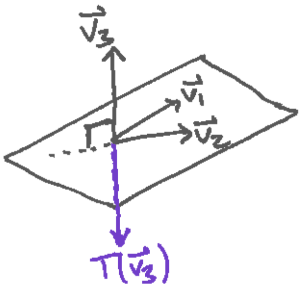
$\iff \vec{x}$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_{n-1}$

$\iff \vec{x} \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{n-1}\}$.

So $\ker(T) = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{n-1}\}$

5. For each of the following statements, determine whether it is always, sometimes or never true.

(a) The determinant of a matrix describing reflection through a plane in $\mathbb{R}^3$ is equal to 1.



Never! Let $\mathcal{B} = \vec{v}_1, \vec{v}_2, \vec{v}_3$ be a basis of $\mathbb{R}^3$ such that $\vec{v}_1, \vec{v}_2$ are in the plane & $\vec{v}_3$ is orthogonal to the plane. Then $T(\vec{v}_1) = \vec{v}_1$, $T(\vec{v}_2) = \vec{v}_2$, $T(\vec{v}_3) = -\vec{v}_3$

$\Rightarrow$ the $\mathcal{B}$-matrix of T is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\Rightarrow \det(\text{mx of } T) = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1 \neq 1.$

(b) Let A and B be $n \times n$ matrices. Then $(A^T B)^T = B^T A$.

Always! Write $(M)_{ij}$ for the $(i,j)^{th}$ entry of M. Then:

$$((A^TB)^T)_{ij} = (A^TB)_{ji} = \sum_{k=1}^{n} (A^T)_{jk} B_{ki}$$
$$= \sum_{k=1}^{n} A_{kj} (B^T)_{ik}$$
$$= \sum_{k=1}^{n} (B^T)_{ik} A_{kj}$$
$$= (B^TA)_{ij} \Rightarrow (A^TB)^T = B^TA.$$

(c) Let A be an $m \times n$ matrix. Then $\det(A^T A) = \det(AA^T)$.

Sometimes!

If $m = n$ then let $A = I_n$

$\Rightarrow \det(A^TA) = 1 = \det(AA^T)$

But if $m = 2$, $n = 3$ and $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ then

$$A^TA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \det(A^TA) = 0$$
$$\neq$$
$$AA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \det(AA^T) = 1$$