Math 290-1 Class 19

Friday 9th November 2018

Properties of determinants

Multilinearity. The determinant is linear in each row and each column. For example:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{nn} \\ \vdots & \vdots & & \vdots \\ ka_{i1}+\ell b_{i1} & ka_{i2}+\ell b_{i2} & \cdots & ka_{in}+\ell b_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \cdot \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{nn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \ell \cdot \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{nn} \\ \vdots & \vdots & & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Alternation. If you swap any two rows in a matrix, its determinant is negated:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{nn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{nn} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Multiplicity. The determinant of a product of matrices is equal to the product of their determinants:

$$\det(AB) = \det(A) \cdot \det(B)$$

Scaling. If A is an $n \times n$ matrix and λ is a scalar, then $\det(\lambda A) = \lambda^n \cdot \det(A)$.

Transpose. The *transpose* A^T of a matrix A is obtained by flipping it across its diagonal: the columns of A^T are the rows of A, and the rows of A^T are the columsn of A. Precisely, the $(i, j)^{\text{th}}$ entry of A^T is a_{ji}. A matrix and its transpose have equal determinant:

$$\det(A^T) = \det(A)$$

Some consequences of all of this include:

- If A is invertible, then $\det(A^{-1}) = 1/\det(A)$.
- If A and B are similar, then $\det(A) = \det(B)$.
- $\det(A^k) = (\det A)^k$ for all k.
- If you add a multiple of a row to another row, the determinant doesn't change.

1. Use elementary row and column operations to find the determinant of the following matrix.

$$\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

2. Use elementary row and column operations to find determinant of the following matrix.

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 1 & 6 & 4 & 8 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & 4 & 12 \end{pmatrix}$$

3. [Bretscher §6.2 Q15] Given that A is a 4×4 matrix with determinant 8, find the determinant of the matrix given by

$$\begin{pmatrix} \cdots & \vec{v}_1 & \cdots \\ \cdots & \vec{v}_1 + \vec{v}_2 & \cdots \\ \cdots & \vec{v}_1 + \vec{v}_2 + \vec{v}_3 & \cdots \\ \cdots & \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 & \cdots \end{pmatrix}$$

where $\vec{v}_i$ is the i^{th} row of A.

4. [Bretscher §6.2 Q48] Let $T : \mathbb{R}^n \to \mathbb{R}$ be the linear transformation defined by

$$T(\vec{x}) = \det \begin{pmatrix} \vdots & \vdots & & \vdots & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \vec{x} \\ \vdots & \vdots & & \vdots & \vdots \end{pmatrix}$$

where $\vec{v}_1, \ldots, \vec{v}_{n-1}$ are linearly independent column vectors in $\mathbb{R}^n$. Describe the kernel of T.

5. For each of the following statements, determine whether it is always, sometimes or never true.

(a) The determinant of a matrix describing reflection through a plane in $\mathbb{R}^3$ is equal to 1.

(b) Let A and B be $n \times n$ matrices. Then $(A^T B)^T = B^T A$.

(c) Let A be an $m \times n$ matrix. Then $\det(A^T A) = \det(AA^T)$.