Math 290-1 Class 18

Wednesday 7th November 2018

Determinants

The **determinant** $\det(A)$ of an $n \times n$ matrix A is an algebraically defined quantity that encodes information about the matrix. In this course, we will primarily use determinants for finding out whether a matrix is invertible and for computing its *eigenvalues* (coming soon); but determinants will reappear in the Spring as a tool in multivariable integral calculus.

The (i, j)-**minor** of an $n \times n$ matrix A is the $(n-1) \times (n-1)$ matrix A_{ij} obtained by deleting the i^{th} row and the j^{th} column. [Circle the $(i, j)^{\text{th}}$ entry in A and delete the row and column that it is in.]

For example, we find the $(3,2)$-minor of the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & \boxed{8} & 9 \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} 1 & \star & 3 \\ 4 & \star & 6 \\ \star & \star & \star \end{pmatrix} \quad \rightsquigarrow \quad A_{32} = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$$

If $A = (a)$ is a 1×1 matrix, then $\det(A) = a$. Otherwise, to find $\det(A)$:

Step 1. Pick a row or column of the matrix to 'expand' along.

Step 2. Find $\det(A_{ij})$ for each position (i, j) in the chosen row or column.

Step 3. Multiply $\det(A_{ij})$ by a_{ij} or $-a_{ij}$ according to the following pattern:

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix} \quad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \quad \begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix} \quad \begin{pmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{pmatrix} \quad \cdots$$

Step 4. Add the results together.

Formally, we have

$$\det(A) \;=\; \underbrace{\sum_{j=1}^{n} (-1)^{i+j}\, a_{ij} \det(A_{ij})}_{\text{expansion along row } i} \;=\; \underbrace{\sum_{i=1}^{n} (-1)^{i+j}\, a_{ij} \det(A_{ij})}_{\text{expansion along column } j}$$

Fun facts about determinants:

- An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.
- The quantity $|\det(A)|$ is a measure of how area (in $\mathbb{R}^2$) or column (in $\mathbb{R}^3$) is scaled by the linear transformation whose matrix is A.

1. Find the determinants of the following matrices.

(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(b) $\begin{pmatrix} 3 & -5 & 8 \\ 0 & 0 & -11 \\ 2 & -1 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 18 & 13 & 30 & 87 & 89 \\ 62 & 88 & 19 & 43 & 22 \\ 96 & 13 & 14 & 44 & 65 \\ 18 & 13 & 30 & 87 & 89 \\ 44 & 46 & 26 & 19 & 87 \end{pmatrix}$

2. Find the volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, where

$$\vec{a} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \qquad \vec{c} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

3. For which values of k is the following matrix non-invertible?

$$\begin{pmatrix} 1 & 2 & -1 \\ k & 0 & 1 \\ 0 & k & 1 \end{pmatrix}$$

4. Explain why the determinant of an upper-triangular matrix is the product of its diagonal entries:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = a_{11} \times a_{22} \times \cdots \times a_{nn}$$