Math 290-1 Class 17

Monday 5th November 2018

Summary of coordinates and change of basis

Let V be an r-dimensional subspace of $\mathbb{R}^n$ and let $\mathfrak{B} = \vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ be a basis of V.

- The **coordinate vector** of a vector $\vec{a}$ in V is defined by

$$[\vec{a}]_{\mathfrak{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{pmatrix} \qquad \text{where} \quad \vec{a} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r$$

  Note that the coefficients $c_1, c_2, \dots, c_r$ are uniquely determined since $\mathfrak{B}$ is a basis of V.

- The **transition matrix** of $\mathfrak{B}$ is the $n \times r$ matrix S whose i^{th} column is $\vec{v}_i$.

$$S = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

  The transition matrix 'decodes' $\mathfrak{B}$-coordinates—that is, $S[\vec{a}]_{\mathfrak{B}} = \vec{a}$ for all $\vec{a}$ in V.

Now we focus on the case when $V = \mathbb{R}^n$, so that $r = n$, and the transition matrix S is an **invertible** $n \times n$ matrix (since its columns form a basis of $\mathbb{R}^n$).

- A vector $\vec{a}$ and its $\mathfrak{B}$-coordinate vector $[\vec{a}]_{\mathfrak{B}}$ are related by

$$\boxed{\vec{a} = S[\vec{a}]_{\mathfrak{B}}} \qquad \text{and} \qquad \boxed{[\vec{a}]_{\mathfrak{B}} = S^{-1}\vec{a}}$$

  In particular, the map $x \mapsto [\vec{x}]_{\mathfrak{B}}$ is a linear transformation.

- The **$\mathfrak{B}$-matrix** (or **matrix with respect to** $\mathfrak{B}$) of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is the matrix B satisfying

$$B[\vec{x}]_{\mathfrak{B}} = [T(\vec{x})]_{\mathfrak{B}}$$

  Intuitively, B is the matrix of T in a world where the vectors in $\mathfrak{B}$ are treated like standard basis vectors. Letting A be the (standard) matrix of T, we have

$$B = \begin{pmatrix} \vdots & \vdots & & \vdots \\ [T(\vec{v}_1)]_{\mathfrak{B}} & [T(\vec{v}_2)]_{\mathfrak{B}} & \cdots & [T(\vec{v}_n)]_{\mathfrak{B}} \\ \vdots & \vdots & & \vdots \end{pmatrix} = S^{-1}AS$$

  and likewise $A = SBS^{-1}$.

- Matrices A and B such that $B = SAS^{-1}$ for some invertible matrix S are called **similar**.

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by orthogonal projection onto the plane whose equation is $x - y - z = 0$.

(a) Find a basis of $\mathbb{R}^3$ relative to which the matrix of T is diagonal.

(b) Find the (standard) matrix of T.

2. For each of the following, determine whether it is true or false.

(a) The matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ are similar for all scalars a, b, c, d.

(b) There is a basis $\mathfrak{B}$ of $\mathbb{R}^2$ relative to which the matrix of the linear transformation whose standard matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is diagonal.