1. Find the matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies

$$T\begin{pmatrix}2\\2\\-1\end{pmatrix} = \begin{pmatrix}0\\3\\-1\end{pmatrix}, \quad T\begin{pmatrix}0\\3\\-1\end{pmatrix} = \begin{pmatrix}2\\2\\-1\end{pmatrix}, \quad T\begin{pmatrix}-1\\-5\\2\end{pmatrix} = \begin{pmatrix}-1\\-5\\2\end{pmatrix}$$

[*Hint:* First find the $\mathfrak{B}$-matrix of T where $\mathfrak{B}$ is a cleverly chosen basis.]

Let $\vec{v}_1 = \begin{pmatrix}2\\2\\-1\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}0\\3\\-1\end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix}-1\\-5\\2\end{pmatrix}$, $\mathfrak{B} = \vec{v}_1, \vec{v}_2, \vec{v}_3$

The $\mathfrak{B}$-matrix of T is $\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} =: B$

The transition matrix of $\mathfrak{B}$ is $\begin{pmatrix}2&0&-1\\2&3&-5\\-1&-1&2\end{pmatrix} =: S$

$\Rightarrow$ the matrix of T is $A = SBS^{-1}$.

Computing S^{-1}:

$$\left(\begin{array}{ccc|ccc}2&0&-1&1&0&0\\2&3&-5&0&1&0\\-1&-1&2&0&0&1\end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|ccc}1&0&0&1&1&3\\0&1&0&1&3&8\\0&0&1&1&2&6\end{array}\right)$$

[See Class 11 solutions for the steps — this was Q3(a).]

So $S^{-1} = \begin{pmatrix}1&1&3\\1&3&8\\1&2&6\end{pmatrix}$

$$\Rightarrow A = SBS^{-1} = \begin{pmatrix}2&0&-1\\2&3&-5\\-1&-1&2\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&3\\1&3&8\\1&2&6\end{pmatrix}$$

$$= \begin{pmatrix}2&0&-1\\2&3&-5\\-1&-1&2\end{pmatrix}\begin{pmatrix}1&3&8\\1&1&3\\1&2&6\end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix}1&4&10\\0&-1&-5\\0&0&1\end{pmatrix}}}$$

2. For each of the following statements, determine whether it is true or false.

(a) If the $\mathfrak{B}$-matrix of A is invertible, then A is invertible.

<u>True!</u> $A = SBS^{-1}$ where S is the transition matrix of $\mathfrak{B}$

$\Rightarrow$ A is a product of invertible matrices

$\Rightarrow$ A is invertible

(b) If A and B are similar $n \times n$ matrices, then there are bases $\mathfrak{B}$ and $\mathfrak{C}$ of $\mathbb{R}^n$ such that B is the $\mathfrak{B}$-matrix of A and A is the $\mathfrak{C}$-matrix of B.

<u>True!</u> Let S be an invertible matrix s.t. $A = SBS^{-1}$.

Then $\mathfrak{B} =$ (columns of S) is a basis $\because$ S is invertible

& B is the $\mathfrak{B}$-matrix of A $\because$ $A = SBS^{-1}$

And $\mathfrak{C} =$ (columns of S^{-1}) is a basis $\because$ S^{-1} is invertible

& A is the $\mathfrak{C}$-matrix of B $\because$ $B = S^{-1}AS = S^{-1}A(S^{-1})^{-1}$.

This is horrendously <u>FALSE</u> in general, e.g. $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ commute but are <u>not</u> similar (by part (a)!)

(c) If matrices A and B commute, then A and B are similar.

~~→ If A & B are invertible then this is true!~~

~~→ If one or both of A & B is not invertible then this could be false or true — eg/ it's true if $A = B = \begin{pmatrix}0&1\\0&0\end{pmatrix}$ $\because$ $A = I_2 B I_2^{-1}$ & $AB = BA = \begin{pmatrix}0&0\\0&0\end{pmatrix}$; it's false if $A = \begin{pmatrix}0&1\\0&0\end{pmatrix}$ and $B = \begin{pmatrix}1&0\\0&1\end{pmatrix}$.~~

(d) For all vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$, all scalars a and b, and all bases $\mathfrak{B}$, we have

$$[a\vec{x} + b\vec{y}]_{\mathfrak{B}} = a[\vec{x}]_{\mathfrak{B}} + b[\vec{y}]_{\mathfrak{B}}$$

<u>True!</u> The function
$$\mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\vec{x} \longmapsto [\vec{x}]_{\mathfrak{B}}$$
is a linear transformation $\because$ $[\vec{x}]_{\mathfrak{B}} = S^{-1}\vec{x}$ where S is the transition matrix of $\mathfrak{B}$.

3. Let $\mathfrak{B} = \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a basis of $\mathbb{R}^n$, let $\mathfrak{C} = \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ be a basis of $\mathbb{R}^m$, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. In terms of the (standard) matrix A of T, find a matrix Q such that

$$Q[\vec{x}]_{\mathfrak{B}} = [T(\vec{x})]_{\mathfrak{C}}$$

Let $R = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \vdots & \vdots & & \vdots \end{pmatrix}$ be the transition matrix of $\mathfrak{B}$

& $S = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_n \\ \vdots & \vdots & & \vdots \end{pmatrix}$ be the transition matrix of $\mathfrak{C}$

$\Rightarrow [\vec{x}]_{\mathfrak{B}} = R^{-1}\vec{x}$ & $[T(\vec{x})]_{\mathfrak{C}} = S^{-1}(T(\vec{x}))$

Then we need

$$\left.\begin{aligned} & Q[\vec{x}]_{\mathfrak{B}} = [T(\vec{x})]_{\mathfrak{C}} \\ \Longleftrightarrow\ & QR^{-1}\vec{x} = S^{-1}(T(\vec{x})) \\ \Longleftrightarrow\ & QR^{-1}\vec{x} = S^{-1}A\vec{x} \end{aligned}\right\} \text{ for all } \vec{x} \text{ in } \mathbb{R}^n$$

$\Longleftrightarrow QR^{-1} = S^{-1}A$

$\Longleftrightarrow Q = \underline{\underline{S^{-1}AR}}$