Math 290-1 Class 15

Wednesday 31st October 2018

Coordinates

Recall that every vector in $\mathbb{R}^n$ can be expressed uniquely a linear combination of the standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$, and the components of the vector give the coefficients of the standard basis vectors in the linear combination. For example, in $\mathbb{R}^3$:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$$

More generally, for any subspace V of $\mathbb{R}^n$ with basis $\mathfrak{B} = \vec{v}_1, \dots, \vec{v}_r$, every vector $\vec{a}$ in V can be expressed uniquely as $c_1\vec{v}_1 + \cdots + c_r\vec{v}_r$, for scalars $c_1, \dots, c_r$. We can think of $c_1, \dots, c_r$ as being the **coordinates** of $\vec{a}$ with respect to the basis $\mathfrak{B}$.

If $\vec{a} = c_1\vec{v}_1 + \cdots + c_r\vec{v}_r$, then write $[\vec{a}]_{\mathfrak{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix}$. Thus the i^{th} component of $[\vec{a}]_{\mathfrak{B}}$ tells you the coefficient of $\vec{v}_i$ in the linear combination, rather than the coefficient of $\vec{e}_i$ (as is usually the case).

Note that $[\vec{a}]_{\mathfrak{B}}$ is the solution to the system

$$\begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \\ \vdots & \vdots & & \vdots \end{pmatrix} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{pmatrix}}_{[\vec{a}]_{\mathfrak{B}}} = \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}}_{\vec{a}}$$

Example

The plane $x + y + z = 0$ in $\mathbb{R}^3$ has basis $\mathfrak{B} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Let $\vec{a} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$. Then $\vec{a} = 3\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, and so $[\vec{a}]_{\mathfrak{B}} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

This example illustrates that if $\vec{a}$ is a vector in an r-dimensional subspace V of $\mathbb{R}^n$, then $[\vec{a}]_{\mathfrak{B}}$ has r components even though $\vec{a}$ itself has n components.

In the case when $V = \mathbb{R}^n$, any basis $\mathfrak{B}$ has n vectors, meaning that both $\vec{a}$ and $[\vec{a}]_{\mathfrak{B}}$ both have n components—**be very careful** not to confuse $\vec{a}$ with $[\vec{a}]_{\mathfrak{B}}$ in this case!

1. For each of the following specifications of a subspace V of $\mathbb{R}^n$, basis $\mathfrak{B}$ of V and vector $\vec{a}$ in V, find the coordinate vector $[\vec{a}]_{\mathfrak{B}}$.

(a) $V = \mathbb{R}^3, \quad \mathfrak{B} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}.$

(b) $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbb{R}^3 : 3x + y = 0 \right\}, \quad \mathfrak{B} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}.$

Parts (c) and (d) are on the next page...

(c) $V = \mathbb{R}^3, \quad \mathfrak{B} = \vec{e}_1, \vec{e}_2, \vec{e}_3, \quad \vec{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$

(d) $V = \mathbb{R}^3, \quad \mathfrak{B} = \vec{e}_3, \vec{e}_1, \vec{e}_2, \quad \vec{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$

2. Find $\vec{a}$ given that $[\vec{a}]_{\mathfrak{B}} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 3 \end{pmatrix}$, where $\mathfrak{B} = \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -7 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 1 \\ 3 \\ 0 \end{pmatrix}.$