1. Find the dimensions of the following subspaces.

(a) The plane in $\mathbb{R}^3$ defined by $2x-y+z=0$.

$$2x-y+z = \begin{pmatrix}2\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2 & -1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

So $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : 2x-y+z=0\right\} = \ker\begin{pmatrix}2 & -1 & 1\end{pmatrix}$

The matrix $(2\ -1\ 1)$ is a 1×3 matrix with rank 1

$\Rightarrow \dim(\text{plane}) = 3 - \dim(\operatorname{im}(2\ -1\ 1)) = 3-1 = 2$

(b) The image of the linear transformation $T:\mathbb{R}^3\to\mathbb{R}^4$ whose matrix is $\begin{pmatrix}2 & 1 & 3\\ 1 & 2 & 0\\ 0 & -1 & 1\\ -1 & 1 & -3\end{pmatrix}$.

$\dim(\operatorname{im} T) = \operatorname{rank}(A)$ where $A =$

$$\begin{pmatrix}2 & 1 & 3\\ 1 & 2 & 0\\ 0 & -1 & 1\\ -1 & 1 & -3\end{pmatrix} \xrightarrow[(IV)+(II)]{(I)-2(II)} \begin{pmatrix}0 & -3 & 3\\ 1 & 2 & 0\\ 0 & -1 & 1\\ 0 & 3 & -3\end{pmatrix} \xrightarrow[(IV)+3(III)]{(I)-3(III)} \begin{pmatrix}0 & 0 & 0\\ 1 & 2 & 0\\ 0 & -1 & 1\\ 0 & 0 & 0\end{pmatrix} \xrightarrow[\text{rows}]{\text{rearrange}} \begin{pmatrix}1 & 2 & 0\\ 0 & -1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$\operatorname{rank}(A) = 2 \Rightarrow \dim(\operatorname{im} T) = 2$.

(c) The kernel of the linear transformation T from part (b).

$$\underbrace{\dim(\operatorname{im} T)}_{=\,2} + \dim(\ker T) = \dim(\operatorname{dom} T) = 3$$

$\Rightarrow \dim(\ker T) = 1$

(d) The set of solutions $\vec{x}$ in $\mathbb{R}^n$ to the equation $a_1x_1+a_2x_2+\cdots+a_nx_n=0$.

Like in (a), $\{\vec{x} \text{ in } \mathbb{R}^n : a_1x_1+\cdots+a_nx_n=0\}$

$= \ker\begin{pmatrix}a_1 & a_2 & \cdots & a_n\end{pmatrix}$

$1\times n$ matrix with rank 1^{**}

$\Rightarrow \dim(\{\vec{x} : a_1x_1+\cdots+a_nx_n=0\})$

$= \dim\ker(a_1\ a_2\ \cdots\ a_n)$

$= n - \dim\operatorname{im}(a_1\ a_2\ \cdots\ a_n)$

$= n - \operatorname{rank}(a_1\ a_2\ \cdots\ a_n)$

$= \boxed{n-1}$

(** if all $a_i = 0$ then it actually has rank 0 $\rightsquigarrow \mathbb{R}^n = \text{kernel} \Rightarrow \dim = n$)

2. For each of the following statements, fill in the sign ($<$, $\leqslant$, $=$, $\geqslant$, $>$) or write **?** if none of these apply.

(a) If U is a subspace of $\mathbb{R}^n$, then $\dim(U)$ $\boxed{\leqslant}$ n.

If $\vec{v}_1 \dots \vec{v}_r$ is a basis of U then $\vec{v}_1 \dots \vec{v}_r$ are linearly independent vectors in $\mathbb{R}^n$ $\Rightarrow r \leq \dim(\mathbb{R}^n) = n$

(b) If A and B are matrices, then $\operatorname{rank}(AB)$ $\boxed{\leqslant}$ $\operatorname{rank}(A)$[redacted].

$\operatorname{im}(AB)$ is contained in $\operatorname{im}(A)$, so any LI set of vectors $\vec{v}_1 \dots \vec{v}_r$ in $\operatorname{im}(AB)$ is also in $\operatorname{im}(A) \Rightarrow r \leq \underset{\text{rank } A}{\underset{\shortparallel}{\dim(\operatorname{im} A)}}$

but $\operatorname{rank}(I_n 0) = \operatorname{rank}(0) = 0 < \operatorname{rank}(I_2) = 2$, so equality need not hold.

(c) If A is an $n \times n$ matrix, then $\operatorname{rank}(A+I)$ $\boxed{?}$ $\operatorname{rank}(A)$.

$\operatorname{rank}(I_2 + I_2) = \operatorname{rank}(2I_2) = 2 = \operatorname{rank}(I_2)$

$\operatorname{rank}((-I_2) + I_2) = \operatorname{rank}(0) = 0 < \operatorname{rank}(-I_2)$

(d) If $m \leq n$ and V is the subspace of $\mathbb{R}^n$ consisting of all solutions $\vec{x}$ to the linear system below, then $\dim(V)$ $\boxed{\geqslant}$ $n-m$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad \vdots \qquad\quad \vdots \qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

Let A be the coefficient matrix of the system (it's an $m \times n$ matrix). Then $\ker(A)$ is the set of solutions to the system ($\equiv A\vec{x} = \vec{0}$)

$$\begin{aligned} \Rightarrow \dim(V) &= \dim(\ker A) \\ &= n - \dim(\operatorname{im} A) \\ &\geqslant n - m \quad (\text{since } \operatorname{rank}(A) \leq \#\text{cols of } A = m) \end{aligned}$$