1. For each of the following subsets of $\mathbb{R}^n$, determine whether or not it is a subspace.

(a) The first quadrant, $Q_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbb{R}^2 : x \geqslant 0 \text{ and } y \geqslant 0 \right\}$

Not a subspace: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is in Q_1 but $\begin{pmatrix} -1 \\ -1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not

(b) The set $P = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{a} \cdot \vec{x} = \vec{b} \cdot \vec{x} = 0\}$, where $\vec{a}$ and $\vec{b}$ are two vectors

P is a subspace:

- $\vec{0}$ is in P : $\vec{a} \cdot \vec{0} = \vec{b} \cdot \vec{0} = 0$
- If $\vec{x}, \vec{y}$ are in P & k, l are scalars, then

$$\vec{a} \cdot (k\vec{x} + l\vec{y}) = k\vec{a} \cdot \vec{x} + l\vec{a} \cdot \vec{y} = k0 + l0 = 0$$

$$\& \ \vec{b} \cdot (k\vec{x} + l\vec{y}) = k\vec{b} \cdot \vec{x} + l\vec{b} \cdot \vec{y} = k0 + l0 = 0$$

$\Rightarrow k\vec{x} + l\vec{y}$ is in P.

(c) The real plane $\mathbb{R}^2$ with both axes (except the origin) removed

Not a subspace: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are in the subset, but $\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is not.

2. (a) Find the matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ whose kernel is spanned by the vectors $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix}$.

Let A = matrix of T. (A is a 4×3 matrix.)

(# free variables in solution to $A\vec{x} = \vec{0}$) $= 2$

$\Rightarrow$ rank(A) = (# variables) − (# free variables) $= 3 - 2 = 1$

So let $A = \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then

$$\begin{cases} A\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow 2a + b - c = 0 \\ A\begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow -3a - c = 0 \end{cases} \xRightarrow{(I)-(II)} 5a + b = 0$$

Let $\underline{a = 1}$, $\underline{b = -5}$, $\underline{c = -3}$. Then $\ker\begin{pmatrix} 1 & -5 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix}\right\}$

(b) Show that the vectors $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$, $\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix}$ are linearly *dependent*.

Let $A = \begin{pmatrix} 1 & -2 & -4 \\ 3 & -2 & 0 \\ -2 & 1 & -1 \end{pmatrix}$.

Row reducing:

$$\begin{pmatrix} 1 & -2 & -4 \\ 0 & 4 & 12 \\ 0 & -3 & -9 \end{pmatrix} \begin{matrix} (II) - 3(I) \\ (III) + 2(I) \end{matrix}$$

$$\begin{pmatrix} 1 & -2 & -4 \\ 0 & 12 & 36 \\ 0 & -12 & -36 \end{pmatrix} \begin{matrix} 3 \times (II) \\ 4 \times (III) \end{matrix}$$

$$\begin{pmatrix} 1 & -2 & -4 \\ 0 & 12 & 36 \\ 0 & 0 & 0 \end{pmatrix} \; (III) + (II)$$

rank$(A) = 2 <$ # columns of A

$\Rightarrow A\vec{x} = \vec{0}$ has infinitely many solutions

$\Rightarrow \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix}$ are linearly dependent

(c) Find a basis for the image of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_1 \\ 2x_3 - x_4 \\ x_4 - 2x_3 \end{pmatrix}$$

$T(\vec{x}) = A\vec{x}$, where $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -2 & 1 \end{pmatrix}$

$\Rightarrow \quad \mathrm{im}(A) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

Since $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 2 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

we have $\mathrm{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\} = \mathrm{im}(A)$

The vectors $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ are linearly independent:

If $k_1\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ then

$$\begin{cases} -k_1 = 0 \\ k_1 = 0 \\ -k_2 = 0 \\ k_2 = 0 \end{cases} \quad \Rightarrow \quad k_1 = k_2 = 0$$

Since $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ span $\mathrm{im}(A)$ and they are linearly independent, they form a basis of $\mathrm{im}(A)$.

3. For each of the following statements, determine whether it is always, sometimes or never true.

(a) A list $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ of vectors in $\mathbb{R}^3$ is linearly independent.

<u>Never.</u> Let $A = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$. Then

$A\vec{x} = \vec{0}$ has infinitely many solutions, since

$\mathrm{rank}(A) \leq (\#\text{ rows of } A) = 3 < 4 = (\#\text{ cols of } A) = (\#\text{ variables})$

$\Rightarrow$ # free variables > 0

$\Rightarrow \ker(A) \neq \{\vec{0}\} \Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly dependent.

(b) The columns of an $m \times n$ matrix A of rank n form a basis im(A).

<u>Always.</u> If $A = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \vdots & \vdots & & \vdots \end{pmatrix}$

then $\mathrm{im}(A) = \mathrm{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

& $\mathrm{rank}(A) = \#\text{ cols of } A = n$

$\Rightarrow A\vec{x} = \vec{0}$ has a unique solution (no free variables)

$\Rightarrow \ker(A) = \{\vec{0}\}$.

$\Rightarrow \vec{v}_1, \dots, \vec{v}_n$ are linearly independent.

(c) The vectors $\begin{pmatrix} 1 \\ k \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1+k \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1+k \end{pmatrix}$ form a basis of $\mathbb{R}^3$.

Let $A = \begin{pmatrix} 1 & 1+k & 1 \\ k & 2 & 0 \\ 0 & 0 & 1+k \end{pmatrix}$ Then they are a basis

$\Leftrightarrow \mathrm{im}(A) = \mathbb{R}^3$ and $\ker(A) = \{\vec{0}\} \Leftrightarrow A$ is invertible.

Putting A in (almost) ref:

$$\begin{pmatrix} 1 & 1+k & 1 \\ k & 2 & 0 \\ 0 & 0 & 1+k \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1+k & 1 \\ 0 & 2-k-k^2 & -k \\ 0 & 0 & 1+k \end{pmatrix} \qquad (II) - k(I)$$

- If $k = 0$ then $2-k-k^2 = 2$ & $1+k = 1 \Rightarrow \mathrm{rank}(A) = 3$
  $\Rightarrow A$ is invertible
  $\Rightarrow$ basis
- If $k = -1$ then $1+k = 0 \Rightarrow \mathrm{rank}(A) < 3 \Rightarrow A$ not invertible
  $\Rightarrow$ not a basis

$\rightsquigarrow$ <u>Sometimes.</u>