Math 290-1 Class 12

Wednesday 24th October 2018

Recap: subspaces, kernels and images

Recall that a **subspace** of $\mathbb{R}^n$ is a set U of n-dimensional vectors which is closed under linear combinations (and contains $\vec{0}$).

- The **kernel** of $T : \mathbb{R}^n \to \mathbb{R}^m$ is the subspace of $\mathbb{R}^n$ given by the vectors $\vec{x}$ for which $T(\vec{x}) = \vec{0}$.

  $$\ker(T) = \{\vec{x} \text{ in } \mathbb{R}^n : T(\vec{x}) = \vec{0}\}$$

  The kernel of a matrix A can be found by solving $A\vec{x} = \vec{0}$.

- The **image** of $T : \mathbb{R}^n \to \mathbb{R}^m$ is the subspace of $\mathbb{R}^m$ given by the values $T(\vec{x})$ of T.

  $$\text{im}(T) = \{\vec{y} \text{ in } \mathbb{R}^m : \vec{y} = T(\vec{x}) \text{ for some } \vec{x}\}$$

  The image of a matrix A is given by the span of the columns of A.

Bases

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ be vectors in a subspace U of $\mathbb{R}^n$.

- The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ **span** U if $\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r) = U$.

  This means that every vector in U is a linear combination of the vectors $\vec{v}_i$. Equivalently:

  $$\text{im}\begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \\ \vdots & \vdots & & \vdots \end{pmatrix} = U$$

- The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ are **linearly independent** if

  $$k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r = \vec{0} \quad \text{implies} \quad k_1 = k_2 = \cdots = k_r = 0$$

  This means that no $\vec{v}_i$ is a linear combination of the other vectors in the list. Equivalently:

  $$\ker\begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \\ \vdots & \vdots & & \vdots \end{pmatrix} = \{\vec{0}\}$$

- The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ are a **basis** of U if they span U and are linearly independent. A basis can be thought of as a 'minimal spanning set': they span U, but if even one of them is removed, the resulting list no longer spans U.

1. For each of the following subsets of $\mathbb{R}^n$, determine whether or not it is a subspace.

(a) The first quadrant, $Q_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbb{R}^2 : x \geqslant 0 \text{ and } y \geqslant 0 \right\}$

(b) The set $P = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{a} \cdot \vec{x} = \vec{b} \cdot \vec{x} = 0\}$, where $\vec{a}$ and $\vec{b}$ are two vectors

(c) The real plane $\mathbb{R}^2$ with both axes (except the origin) removed

2. (a) Find the matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ whose kernel is spanned by the vectors $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix}$.

(b) Show that the vectors $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$, $\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix}$ are linearly *dependent*.

(c) Find a basis for the image of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_1 \\ 2x_3 - x_4 \\ x_4 - 2x_3 \end{pmatrix}$$

3. For each of the following statements, determine whether it is always, sometimes or never true.

(a) A list $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ of vectors in $\mathbb{R}^3$ is linearly independent.

(b) The columns of an $m \times n$ matrix A of rank n form a basis $\text{im}(A)$.

(c) The vectors $\begin{pmatrix} 1 \\ k \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1+k \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1+k \end{pmatrix}$ form a basis of $\mathbb{R}^3$.