Math 290-1 Class 10

Friday 19th October 2018

Subspaces

A **subspace** of $\mathbb{R}^n$ is a set U of n-dimensional vectors which is closed under linear combinations. Thus if $\vec{u}$ and $\vec{v}$ are in U, then $a\vec{u}+b\vec{v}$ is in U for all scalars a and b. Some examples:

- The subspaces of $\mathbb{R}^2$ are: $\mathbb{R}^2$ itself; any line through the origin; and the point at the origin.
- The subspaces of $\mathbb{R}^3$ are: $\mathbb{R}^3$ itself; any plane through the origin; any line through the origin; and the point at the origin.

Note that the zero vector $\vec{0}$ is in *all* subspaces—it is the linear combination of no vectors at all!

Span, kernel and image

The **span** of a set of vectors is the set of linear combinations of those vectors. For example

$$\operatorname{span}(\vec{u},\vec{v},\vec{w}) = \{a\vec{u}+b\vec{v}+c\vec{w} \mid a,b,c \text{ scalars}\}$$

Spans are subspaces, since any linear combination of linear combinations is a linear combination.

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$:

- The **kernel** of T is the set $\ker(T)$ of solutions $\vec{x}$ in $\mathbb{R}^n$ to the equation $T(\vec{x}) = \vec{0}$.
- The **image** of T is the set $\operatorname{im}(T)$ of vectors $\vec{b}$ in $\mathbb{R}^m$ such that the equation $T(\vec{x}) = \vec{b}$ has a solution. [That is, $\operatorname{im}(T)$ consists of all vectors of the form $T(\vec{x})$.]

For an $m \times n$ matrix A, define $\ker(A)$ and $\operatorname{im}(A)$ analogously. Some remarks:

- The kernel of $T : \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of $\mathbb{R}^n$, and the image of T is a subspace of $\mathbb{R}^m$.
- The image of a matrix A is the span of the columns of A. (Why?)
- The kernel of a matrix A can be found by solving the system $A\vec{x} = \vec{0}$ and expressing the result as a linear combination of constant vectors in $\mathbb{R}^n$—these are the vectors that span the kernel.

Some remarks regarding invertibility:

- An $n \times n$ matrix A is invertible if and only if $\ker(A) = \{\vec{0}\}$—this says that the zero vector is the unique solution to the equation $A\vec{x} = \vec{0}$.
- An $n \times n$ matrix A is invertible if and only if $\operatorname{im}(A) = \mathbb{R}^n$—this says that the equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$ in $\mathbb{R}^n$.

1. Find the kernel and image of each of the following matrices, expressing your answers as a span of as few vectors as you can.

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

(b) $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(c) $C = \begin{pmatrix} 15 & 94 & 43 & 72 & 55 & 13 \\ 0 & 81 & 46 & 46 & 21 & 54 \\ 0 & 0 & 62 & 58 & 32 & 30 \\ 0 & 0 & 0 & 49 & 29 & 11 \\ 0 & 0 & 0 & 0 & 40 & 62 \\ 0 & 0 & 0 & 0 & 0 & 12 \end{pmatrix}$

2. Let $\vec{v} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ be a nonzero vector in $\mathbb{R}^3$. Show that the kernel of the linear transformation $\text{proj}_{\vec{v}} : \mathbb{R}^3 \to \mathbb{R}^3$ is the plane defined by $3x - y + z = 0$, and express this plane as the span of two vectors.