Math 290-1 Class 8

Monday 15th October 2018

Inverse functions: quick facts

A function $f : X \to Y$ is **invertible** if any of the following conditions hold:

- f has an **inverse**, which is a function $f^{-1} : Y \to X$ such that $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$ for all $x \in X$ and $y \in Y$.
- For each $y \in Y$, the equation $f(x) = y$ has a unique solution x (which is then equal to $f^{-1}(y)$).

Some nice properties hold, for example:

- If f is invertible, then so is f^{-1}, and $(f^{-1})^{-1} = f$.
- If $f : X \to Y$ and $g : Y \to Z$ are invertible, so is $g \circ f : X \to Z$, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Inverse matrices

If $m \neq n$ then a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ cannot have an inverse. (Why?) So we only consider linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ and **square** ($n \times n$) matrices.

If $T(\vec{x}) = \vec{y}$ is to have an inverse, then we should be able to solve for $\vec{x}$ and the solution should be *unique*. Writing $A = (a_{ij})$ for the matrix of T and $B = (b_{ij})$ for the matrix of T^{-1}, this corresponds to doing some sequence of algebraic manipulations involving equations to find $T^{-1}(\vec{y})$ as shown:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \quad y_2 \\ \quad\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad\qquad \ddots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \qquad\qquad y_n \end{cases} \rightsquigarrow \begin{cases} x_1 \qquad\qquad = b_{11}y_1 + b_{12}y_2 + \cdots + b_{1n}y_n \\ \quad x_2 \qquad = b_{21}y_1 + b_{22}y_2 + \cdots + b_{2n}y_n \\ \qquad \ddots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \qquad\qquad x_n = b_{n1}y_1 + b_{n2}y_2 + \cdots + b_{nn}y_n \end{cases}$$

These operations involving equations can be represented using elementary row operations of an $n \times n$ matrix augmented by another $n \times n$ matrix:

$$\left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & 1 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{array}\right)$$

The matrix B is called the **inverse** to A, and we write $B = A^{-1}$. Some consequences:

- T is invertible if and only if $\text{rref}(A) = I_n$, the $n \times n$ **identity matrix**.
- T is invertible if and only if $\text{rank}(A) = n$.
- If A is invertible, then A^{-1} can be computed via $\text{rref}\left(A \mid I_n\right) = \left(I_n \mid A^{-1}\right)$
- $AA^{-1} = A^{-1}A = I_n$ and $(A^{-1})^{-1} = A$ and $(BA)^{-1} = A^{-1}B^{-1}$

1. For each of the following matrices, determine whether or not it is invertible and, if it is invertible, find its inverse matrix.

(a) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

2. Find a 3×2 matrix A such that

$$A \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 2 & -3 \end{pmatrix}$$

3. For which values of b and c is the following matrix invertible?

$$\begin{pmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

4. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix.

Prove that A is invertible if and only if $ad = bc$, and that in this case A^{-1} is defined as follows.

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

5. Let A be an *upper-triangular* $n \times n$ matrix, i.e. such that $a_{ij} = 0$ if $i > j$. Prove that A is invertible if and only if $a_{11} \times a_{22} \times \cdots \times a_{nn} \neq 0$.