Math 290-1 Class 7

Friday 12th October 2018

Matrix products

If $S:\mathbb{R}^n \to \mathbb{R}^r$ and $T:\mathbb{R}^r \to \mathbb{R}^m$ are linear transformations, then their composite $T\circ S:\mathbb{R}^n \to \mathbb{R}^m$ is defined by $(T\circ S)(\vec{v}) = T(S(\vec{v}))$ for all $\vec{v}$ in $\mathbb{R}^n$. It turns out that $T\circ S$ is also linear:

$$\boxed{(T\circ S)(\vec{v}+\vec{w})} = T(S(\vec{v}+\vec{w})) = T(S(\vec{v})+S(\vec{w})) = T(S(\vec{v}))+T(S(\vec{w})) = \boxed{(T\circ S)(\vec{v})+(T\circ S)(\vec{w})}$$

$$\boxed{(T\circ S)(k\vec{v})} = T(S(k\vec{v})) = T(kS(\vec{v})) = kT(S(\vec{v})) = \boxed{k(T\circ S)(\vec{v})}$$

If T is represented by the $m\times r$ matrix B and S is represented by the $r\times n$ matrix A, then we write BA for the $m\times n$ matrix which represents the linear transformation $T\circ S$. The matrix BA is called the **matrix product** of B and A.

For $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$, the $(i,j)^{\text{th}}$ entry of BA can be computed as:

$$(BA)_{ij} = \sum_{k=1}^{r} b_{ik}a_{kj} = b_{i1}a_{1j}+b_{i2}a_{2j}+\cdots+b_{ir}a_{rj}$$

In other words, writing $\vec{b}_{i\bullet}$ for the i^{th} *row* of B and $\vec{a}_{\bullet j}$ for the j^{th} *column* of A, we have

$$BA = \begin{pmatrix} \cdots & \vec{b}_{1\bullet} & \cdots \\ \cdots & \vec{b}_{2\bullet} & \cdots \\ & \vdots & \\ \cdots & \vec{b}_{m\bullet} & \cdots \end{pmatrix} \begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{a}_{\bullet 1} & \vec{a}_{\bullet 2} & \cdots & \vec{a}_{\bullet n} \\ \vdots & \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} \vec{b}_{1\bullet}\cdot\vec{a}_{\bullet 1} & \vec{b}_{1\bullet}\cdot\vec{a}_{\bullet 2} & \cdots & \vec{b}_{1\bullet}\cdot\vec{a}_{\bullet n} \\ \vec{b}_{2\bullet}\cdot\vec{a}_{\bullet 1} & \vec{b}_{2\bullet}\cdot\vec{a}_{\bullet 2} & \cdots & \vec{b}_{2\bullet}\cdot\vec{a}_{\bullet n} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{b}_{m\bullet}\cdot\vec{a}_{\bullet 1} & \vec{b}_{m\bullet}\cdot\vec{a}_{\bullet 2} & \cdots & \vec{b}_{m\bullet}\cdot\vec{a}_{\bullet n} \end{pmatrix}$$

[Notice that $\vec{b}_{i\bullet}$ and $\vec{a}_{\bullet j}$ are both r-dimensional vectors, so their dot product is defined.]

Properties of the matrix product

- BA is defined if and only if (# columns of B) = (# rows of A);
- If B is $m\times r$ and A is $r\times n$, then BA is $m\times n$;
- $(CB)A = C(BA)$, so it doesn't matter what order you multiply — this is called *associativity*;
- $D(B+C) = DB+DC$ and $(B+C)A = BA+CA$ — this is called *distributivity*;
- $(kB)A = B(kA) = kBA$ for all scalars k and matrices A,B.

Some warnings about matrix products

- In general $BA\neq AB$, even if both products are defined—we say that A **commutes** with B if $BA = AB$. (Notice that A and B must be square matrices for this to make sense.)
- It's possible for a nonzero matrix A to satisfy $A^2 = 0$ (where 0 is the *zero matrix*).
- You can't *divide* by a matrix (unless it's *invertible*—more of this on Monday).

1. For each of the following products of matrices, either compute its value, or explain why it is not defined.

(a) $\begin{pmatrix} 2 & 2 \\ 1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & 1 \\ -2 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 2 \\ 1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & -2 \\ 1 & 1 & -2 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & 1 \\ -2 & -2 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 23 & 56 & 89 \\ 13 & 24 & 35 \\ 77 & 44 & 11 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

(g) $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

(h) $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$

2. Four 2×2 matrices A, B, C and D are defined as follows. Which pairs commute?

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

3. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates each vector by θ radians anticlockwise about the origin, and let A_θ be the associated 2×2 matrix.

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Using the fact that $R_\alpha \circ R_\beta = R_{\alpha+\beta}$, find expressions for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ in terms of $\sin\alpha$, $\cos\alpha$, $\sin\beta$ and $\cos\beta$.