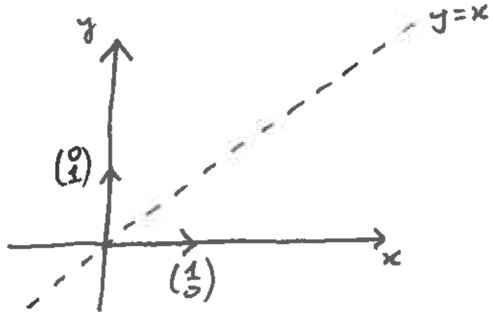
**1.** Find the matrix of the linear transformation $Q : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects each vector through the line $y = x$.

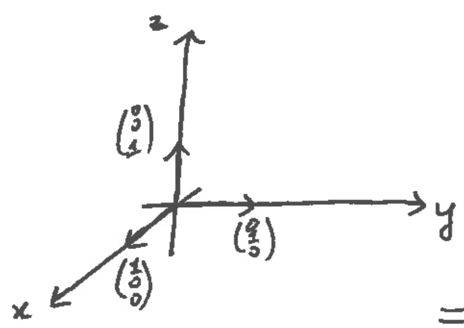



$$Q\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$Q\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$\Rightarrow$ The matrix of $Q$ is $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$.

**2.** Find the matrix of the linear transformation $R : \mathbb{R}^3 \to \mathbb{R}^3$ that scales each vector by a factor of 2 and then reflects it in the $(x, y)$-plane.



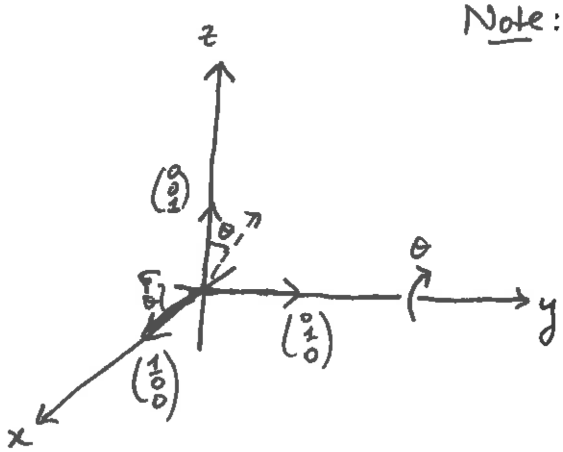

$$R\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\0\\0\end{pmatrix}$$

$$R\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}$$

$$R\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\-2\end{pmatrix}$$

$\Rightarrow$ The matrix of $R$ is $\begin{pmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & -2\end{pmatrix}$

**3.** Find the matrix of the linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates each vector by $\theta$ radians about the $y$-axis.




Note: $y$-axis remains fixed
$(x, z)$-plane maps to $(x, z)$-plane

$$S\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\0\\\sin\theta\end{pmatrix}$$

$$S\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$S\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-\sin\theta\\0\\\cos\theta\end{pmatrix}$$

$\Rightarrow$ So the matrix of $S$ is

$$\begin{pmatrix}\cos\theta & 0 & -\sin\theta\\0 & 1 & 0\\\sin\theta & 0 & \cos\theta\end{pmatrix}$$

**4.** Given a fixed nonzero vector $\vec{a}$, use the following three facts to find an explicit formula for the linear map $\text{proj}_{\vec{a}} : \mathbb{R}^n \to \mathbb{R}^n$.

(i) $\vec{v}^{\parallel} = k\vec{a}$ for some scalar $k$ (since $\vec{v}^{\perp}$ is parallel to $\vec{a}$)

(ii) $\vec{v}^{\perp} \cdot \vec{a} = 0$ (since $\vec{v}^{\perp}$ is perpendicular to $\vec{a}$)

(iii) $\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$

[*Hint: start by writing $\vec{v}^{\perp}$ in terms of $\vec{v}$ and $\vec{v}^{\parallel}$ in equation (ii).*]

(iii) $\Rightarrow \vec{v}^{\perp} = \vec{v} - \vec{v}^{\parallel}$

$\Rightarrow 0 = \vec{v}^{\perp} \cdot \vec{a} = (\vec{v} - \vec{v}^{\parallel}) \cdot \vec{a}$ by (ii)

$\Rightarrow 0 = \vec{v} \cdot \vec{a} - \vec{v}^{\parallel} \cdot \vec{a}$ since dot product is linear

$\Rightarrow \vec{v}^{\parallel} \cdot \vec{a} = \vec{v} \cdot \vec{a}$ ($+ \vec{v}^{\parallel} \cdot \vec{a}$ both sides)

$\Rightarrow (k\vec{a}) \cdot \vec{a} = \vec{v} \cdot \vec{a}$ by (i)

$\Rightarrow k(\vec{a} \cdot \vec{a}) = \vec{v} \cdot \vec{a}$ since dot product is linear

$\Rightarrow k = \dfrac{\vec{v} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}$ ($\div \vec{a} \cdot \vec{a}$ both sides)

Using (i) again, we have

$$\text{proj}_{\vec{a}}(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\right)\vec{a} \quad \text{for all } \vec{v} \text{ in } \mathbb{R}^n$$

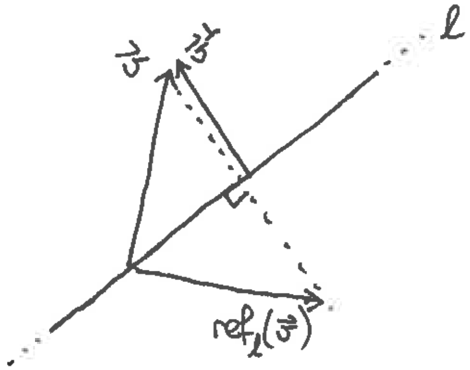
Find the orthogonal projection of $\underbrace{\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}}_{\vec{v}}$ onto the vector $\underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{\vec{a}}$.

$\vec{v} \cdot \vec{a} = 2 \times 1 + 3 \times 0 + (-1) \times 1 = 1$

$\vec{a} \cdot \vec{a} = 1 \times 1 + 0 \times 0 + 1 \times 1 = 2$

$$\Rightarrow \text{proj}_{\vec{a}}(\vec{v}) = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \left( = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \right)$$

**5.** Let $\ell$ be a line through the origin in $\mathbb{R}^n$ which is parallel to a vector $\vec{a}$. Find an expression for the linear transformation $\mathrm{ref}_\ell : \mathbb{R}^n \to \mathbb{R}^n$ that reflects each vector $\vec{v}$ through the line $\ell$.




$$\mathrm{ref}_\ell(\vec{v}) = \vec{v} - 2\vec{v}^{\perp}$$

Note we can express $\mathrm{ref}_\ell$ in terms of $\mathrm{proj}_\ell$:

$$\begin{aligned}
\mathrm{ref}_\ell(\vec{v}) &= \vec{v} - 2\vec{v}^{\perp} && \text{since } \vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp} \\
&= \vec{v} - 2(\vec{v} - \vec{v}^{\parallel}) && \text{since } \vec{v}^{\parallel} = \mathrm{proj}_\ell(\vec{v}) \\
&= 2\,\mathrm{proj}_\ell(\vec{v}) - \vec{v} \\
&= 2\left(\frac{\vec{v}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\right)\vec{a} - \vec{v}
\end{aligned}$$

Find the result of reflecting the vector $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ through the line which passes through the origin and is parallel to the vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

From Q4 we know that $\mathrm{proj}_{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}$

$$\begin{aligned}
\text{So } \mathrm{ref}_\ell \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} &= 2 \cdot \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}.
\end{aligned}$$