Math 290-1 Class 5

Monday 8th October 2018

The **standard basis vectors** for $\mathbb{R}^n$ are the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, where $\vec{e}_i$ is the n-dimensional vector whose i^{th} component is 0 and all other components are 1. For example, the standard basis vectors for $\mathbb{R}^3$ are

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Notice that every vector in $\mathbb{R}^n$ is a linear combination of the standard basis vectors.

$$\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n \qquad \text{e.g.} \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In an $m \times n$ matrix A, the j^{th} column is given by $A\vec{e}_j$ for all $1 \leqslant j \leqslant m$. For example

$$\begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if

(a) $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for all $\vec{v}, \vec{w}$ in $\mathbb{R}^n$; and

(b) $T(k\vec{v}) = kT(\vec{v})$ for all $\vec{v}$ in $\mathbb{R}^n$ and all scalars k.

Conditions (a) and (b) imply that T respects linear combinations:

$$T(k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r) = k_1T(\vec{v}_1) + k_2T(\vec{v}_2) + \cdots + k_rT(\vec{v}_r)$$

Consequently, we arrive at the following result.

Theorem
A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if there is a matrix A such that $T(\vec{v}) = A\vec{v}$ for all vectors $\vec{v}$ in $\mathbb{R}^n$. The j^{th} column of A is given by the vector $T(\vec{e}_j)$.

The main takeaways of this theorem are:

- Linear transformations are 'the same thing' as matrices.
- The matrix associated to a linear transformation can be found by looking where the linear transformation sends the standard basis vectors.

Examples of linear transformations include *scalings*, *rotations* and *reflections*—we'll talk more about the geometric viewpoint on Wednesday.

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **invertible** if the linear system of equations given by $\vec{y} = T(\vec{x})$ has a unique solution for all $\vec{y}$ in $\mathbb{R}^n$. (In fact, we need $m = n$ for T to be invertible!)

1. For each of the following functions T, determine whether or not it is linear. If it is linear, find a matrix A such that $T(\vec{v}) = A\vec{v}$ for all vectors $\vec{v}$.

(a) $T : \mathbb{R}^2 \to \mathbb{R}$, defined by $T\begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1^2 + x_2^2$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_1 + x_2 \end{pmatrix}$

(c) [Bretscher §2.1 Q4] $T : \mathbb{R}^3 \to \mathbb{R}^4$, defined by the linear system

$$\begin{cases} y_1 = 9x_1 + 3x_2 - 3x_3 \\ y_2 = 2x_1 - 9x_2 + x_3 \\ y_3 = 4x_1 - 9x_2 - 2x_3 \\ y_4 = 5x_1 + x_2 + 5x_3 \end{cases}$$

(i.e. T is defined by $\vec{y} = T(\vec{x})$, with $\vec{y}$ defined in terms of $\vec{x}$ according to these four equations)

(d) [Bretscher §2.1 Q4, modified]

$$\begin{cases} y_1 = 9x_1 + 3x_2 - 3x_3 + 2 \\ y_2 = 2x_1 - 9x_2 + x_3 \\ y_3 = 4x_1 - 9x_2 - 2x_3 - 1 \\ y_4 = 5x_1 + x_2 + 5x_3 + 1 \end{cases}$$

2. Given a vector $\vec{a}$ in $\mathbb{R}^n$, find a matrix A such that $A\vec{v} = \vec{a} \cdot \vec{v}$ for all $\vec{v}$ in $\mathbb{R}^n$.

3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be any linear transformation. Find $T(\vec{0})$, where $\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is the *zero vector*.

4. Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates each vector $\vec{v}$ by $\frac{\pi}{3}$ radians about the origin.