Optimization on a compact set

21-259, Tuesday 7th October 2014, Clive Newstead

General set-up

Suppose $f(x,y)$ is a function of two variables and that R is some compact (closed and bounded) subset of the xy-plane. On such sets, it's known that f is bounded and attains its absolute maxima and minima.[1] Our goal is to find them.

The procedure

- **Step 1.** Find the critical points of f which lie in the region R.

  More specifically, find where $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$, or where these don't exist, and make a note of all the points you find.

- **Step 2.** Find the critical points of f on the boundary of the region R.

  More specifically, we can typically describe the boundary of R by relating x and y via an equation. We can then substitute for one or the other of the variables into $f(x,y)$ to obtain a single-variable function. Finding the extreme values on the boundary thus reduces to a univariate calculus problem. Don't forget to check end-points of line segments if there are any!

- **Step 3.** Evaluate the function at all the points you found in Steps 1 and 2. The greatest and least values you obtain are the global maxima and minima, respectively, of f on R.

Example: Worksheet Question 5

On the recitation worksheet you were asked to find the absolute maximum and minimum values of $f(x,y) = x^2 + y^2 - 2x - 4y$ on the triangular region of the xy-plane bounded by the lines $x = 0$, $y = 3$ and $y = x$.

Step 1. To find the critical points we set $f_x = f_y = 0$. The equation $f_x = 0$ gives

$$2x - 2 = 0 \quad \Rightarrow \quad x = 1$$

The equation $f_y = 0$ gives

$$2y - 4 = 0 \quad \Rightarrow \quad y = 2$$

So the only critical point of f is the point $\boxed{(1,2)}$. This lies in the triangular region described above, so we make note of it... say like putting it in a box, like I just did. (If it so happened that this point lay outside of the region, we could discard it.)

[1] This is the two-dimensional extreme value theorem: recall that the one-dimensional version says that if $f(x)$ is a continuous function on a closed and bounded interval $[a,b]$ then it attains its extreme values on $[a,b]$.

Step 2. The boundary of the curve consists of three line segments:

(a) The line $x = 0$, where $0 \leqslant y \leqslant 3$.

(b) The line $y = 3$, where $0 \leqslant x \leqslant 3$.

(c) The line $y = x$, where $0 \leqslant x \leqslant 3$.

(You can find the end-points of the line segments by drawing a picture or by computing where the three boundary lines intersect pairwise.)

In each case we *should* check the end-points, but there'll be some overlap, so just note that the vertices of the triangle are $\boxed{(0,0)}$, $\boxed{(0,3)}$ and $\boxed{(3,3)}$. Now we'll find any critical points lying on the edges of the triangle away from the vertices. Labelling the steps (a)–(c) as above:

(a) Let $g(y) = f(0, y) = y^2 - 4y$. Then $g'(y) = 2y - 4$, which equals zero when $y = 2$. So $\boxed{(0,2)}$ is a critical point on the boundary.

(b) Let $h(x) = f(x, 3) = x^2 - 2x - 3$. Then $h'(x) = 2x - 2$, which equals zero when $x = 1$. So $\boxed{(1,3)}$ is a critical point on the boundary.

(c) Let $k(x) = f(x, x) = 2x^2 - 6x$. Then $k'(x) = 4x - 6$, which equals zero when $x = \frac{3}{2}$. So $\boxed{(\frac{3}{2}, \frac{3}{2})}$ is a critical point on the boundary.

We've now found seven points where we need to find the value of x:

Point (a,b)	$(1,2)$	$(0,0)$	$(0,3)$	$(3,3)$	$(0,2)$	$(1,3)$	$(\frac{3}{2},\frac{3}{2})$
Value $f(a,b)$	-5	0	-3	0	-4	-4	$-\frac{9}{2}$

We thus see that the absolute maximum value of f on the triangular region is 0, which occurs at $(0, 0)$ and $(3, 3)$; and the absolute minimum value is -5, which occurs at $(1, 2)$.

Notice that $(1, 2)$ is the only critical point of f, and (as a quick check shows) it's a local minimum; so if we weren't constrained to the triangular region, f would have no absolute maximum value, but it *would* still have a minimum value of -5 at the point $(1, 2)$.