21-256: Tangent planes and linear approximation

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Tangent planes

Equations involving three variables all describe surfaces in $\mathbb{R}^3$; moreover, any such equation can be rearranged to take the form $f(x, y, z) = 0$, just by subtracting everything from one side of the equation.

Let f be a function of three variables x, y, z. The *tangent plane* to the surface $f(x, y, z) = 0$ at the point (a, b, c) is the plane whose equation is

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0$$

In particular, it's easy to use the above to work out that the tangent plane to the surface $z = g(x, y)$ at the point $(a, b, g(a, b))$ is given by

$$z - g(a, b) = g_x(a, b)(x - a) + g_y(a, b)(y - b)$$

[Recall: $f_x(a, b, c)$ is $\frac{\partial f}{\partial x}$ evaluated when $x = a, y = b, z = c$, and so on.]

Linearization and linear approximation

Consider the case where we have a function $z = g(x, y)$. The tangent plane to the surface $z = f(x, y)$ at a particular point intersects the surface at that point. Thus, when $x \approx a$ and $y \approx b$, the tangent plane is a good approximation to the surface.

The *linearization* of $g(x, y)$ at the point (a, b) is a function $L(x, y)$ defined by

$$L(x, y) = g(a, b) + g_x(a, b)(x - a) + g_y(a, b)(y - b)$$

Notice that the graph of $z = L(x, y)$ is precisely the tangent plane at $(a, b, g(a, b))$.

Hence, when $x \approx a$ and $y \approx b$, we have $L(x, y) \approx g(x, y)$. The value of $L(x, y)$ is called the *linear approximation* of $g(x, y)$.