21-256: Partial differentiation

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This is a summary of the important results about partial derivatives and the chain rule that you should know.

Partial derivatives

Given a function f of n variables, $x_1, x_2, \cdots, x_n$, the *partial derivative* of f with respect to x_i is

$$\lim_{h \to 0} \left(\frac{f(x_1, x_2, \cdots, x_{i-1}, x_i + h, x_{i+1}, \cdots, x_n) - f(x_1, \cdots, x_n)}{h} \right)$$

That is, it's what you get when you differentiate f with respect to x_i, treating all the remaining variables as constants. We write this quantity as

$$\frac{\partial f}{\partial x_i} \qquad \text{or} \qquad f_{x_i}(x_1, \cdots, x_n) \qquad \text{or} \qquad f_i(x_1, \cdots, x_n)$$

For example, if $g(x,y) = x^2 + y^2 \sin x$ then $f_x(x,y) = 2x + y^2 \cos x$ and $f_y(x,y) = 2y \sin x$.

Notes on reversing partial differentiation can be found at `http://math.cmu.edu/~cnewstea/`.

Chain rule

Let f be a function of $x_1, x_2, \cdots, x_n$ and suppose all the x_is are functions of 'new' variables $t_1, t_2 \cdots, t_k$. The *chain rule* states that

$$\frac{\partial f}{\partial t_j} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_j}$$

In particular

- If $f = f(x,y)$ and $x = x(t)$ and $y = y(t)$; that is, the inputs to f depend on a single variable t, then

  $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

  Notice the derivatives with respect to t are usual (total) derivatives because the dependence is only one one variable.

- If $f = f(x,y)$ and $x = x(s,t)$ and $y = y(s,t)$, then

  $$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Examples

(1) Suppose $z = x^2y + 3xy^4$, where $x = \sin(2t)$ and $y = \cos(t)$. Find $\dfrac{dy}{dt}$ when $t = 0$.

Using the chain rule, we have $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$. Now

- $\dfrac{\partial z}{\partial x} = 2xy + 3y^4$ and $\dfrac{\partial z}{\partial y} = x^2 + 12xy^3$;
- When $t = 0$, $\dfrac{dx}{dt} = 2\cos(2t) = 2$ when $t = 0$ and $\dfrac{dy}{dt} = -\sin t = 0$;
- When $t = 0$, $x = \sin(2 \cdot 0) = 0$ and $y = \cos(0) = 1$.

Putting all of this together:

$$\frac{dz}{dt} = (2 \cdot 0 \cdot 1 + 3 \cdot 1^4) \cdot 2 + (\text{stuff}) \cdot 0 = 6$$

(2) If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, then using the chain rule we get

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= (e^x \sin y)(t^2) + (e^x \cos y)(2st) \\ &= te^x(t \sin y + 2s \cos y) \\ &= te^{st^2}(t \sin(s^2t) + 2s\cos(s^2t)\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= (e^x \sin y)(2st) + (e^x \cos y)(s^2) \\ &= se^x(2t \sin y + s \cos y) \\ &= se^{st^2}(2t \sin(s^2t) + s\cos(s^2t)\end{aligned}$$

(3) Suppose $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$ and $z = r^2s\sin(t)$. Find $\dfrac{\partial u}{\partial s}$ when $(r, s, t) = (2, 1, 0)$.

The chain rule tells us that

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} \\ &= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin(t))\end{aligned}$$

When $r = 2$, $s = 1$ and $t = 0$, we have

$$x = 2 \cdot 1 \cdot e^0 = 2, \quad y = 2 \cdot 1^2 \cdot e^{-0} = 2, \quad z = 2^2 \cdot 1 \cdot \sin(0) = 0$$

Putting all this information together gives

$$\frac{\partial u}{\partial s} = \underbrace{(4 \cdot 2^3 \cdot 2)(2e^0)}_{=128} + \underbrace{(2^4 + 2 \cdot 1 \cdot 0^3)(2 \cdot 2 \cdot 1 \cdot e^{-0})}_{=64} + \underbrace{(3 \cdot 1^2 \cdot 0^2)}_{=0} \cdot (\text{stuff}) = 192$$

Reversing partial differentiation

When we partially differentiate with respect to a variable x, we treat all variables other than x as if they were constants. Thus, if we want to reverse the process, unlike in the one-dimensional case when you integrate and 'add a constant', we need to 'add a function of the variables we're treating as constant'.

More precisely:

- In one dimension, differentiation with respect to x is reversed by integrating with respect to x and adding a constant. That is, if $\dfrac{dz}{dx} = f(x)$ and $F(x)$ is a function with $F'(x) = f(x)$, then $z = F(x) + C$, where C is a constant.
- In two dimensions, partial differentiation with respect to x is reversed by integrating with respect to x (treating y as constant) and adding a function of y. That is, if $\dfrac{\partial z}{\partial x} = f(x, y)$ and $F(x, y)$ is a function with $F_x(x, y) = f(x, y)$, then $z = F(x, y) + A(y)$, where A is a function of y.
- Higher dimensions are similar.

Let's see why this makes sense: since $A(y)$ depends only on y, it is treated as a constant when partially differentiated with respect to x, and so $\frac{\partial A}{\partial x} = 0$. Thus

$$\frac{\partial z}{\partial x} \quad = \quad \frac{\partial F}{\partial x} + \frac{\partial A}{\partial x} \quad = \quad F_x(x, y) + 0 \quad = \quad f(x, y)$$

which is what we wanted.

If we integrate treating y as constant, our constant of integration is whatever our integral thinks is constant... but our integral thinks anything not depending on x (i.e. any function of y) is a constant. In summary,

$$\frac{\partial z}{\partial x} = f(x, y) \qquad \Longrightarrow \qquad z = \underbrace{\int f(x, y)\, dx}_{\text{treating } y \text{ as constant}} \; + A(y)$$

Thus we can find an expression for a function if we know its partial derivatives and initial values.

Example 1

Suppose k is a function of the variables s, t, satisfying

$$\frac{\partial k}{\partial s} = 2st, \qquad \frac{\partial k}{\partial t} = s^2 + 3t^2, \qquad k(0,0) = 1;$$

We want to find an expression for $k(s,t)$. We do this in three steps: (1) reverse $\frac{\partial k}{\partial s}$; (2) differentiate what we get with respect to t and compare with $\frac{\partial k}{\partial t}$; (3) sub in the initial values.

(1) We know that $\dfrac{\partial k}{\partial s} = 2st$. Integrating with respect to s and treating t as constant, we have

$$k(s,t) = s^2 t + A(t)$$

for some function A of t. (You can check that this gives $\frac{\partial k}{\partial s} = 2st$, as specified.)

(2) We're given that $\dfrac{\partial k}{\partial t} = s^2 + 3t^2$. Partially differentiating the expression from (1) with respect to t gives

$$\frac{\partial k}{\partial t} = s^2 + 3t^2 = s^2 + A'(t)$$

Subtracting s^2 from each side gives $A'(t) = 3t^2$, and hence $A(t) = t^3 + C$ for a constant C.[1]

(3) Substituting our expression for $A(t)$ into the expression for $k(s,t)$ in (1), we have

$$k(s,t) = s^2 t + t^3 + C$$

for some constant C. We're told that $k(0,0) = 1$, so

$$1 = k(0,0) = 0^2 \cdot 0 + 0^3 + C = C$$

and hence $C = 1$.

In summary,

$$k(s,t) = s^2 t + t^3 + 1$$

[1] Since A was a function of just t alone, C really *is* a constant instead of a function of s. Otherwise A would depend on s, so it wouldn't vanish when we partially differentiate with respect to s.

Example 2

Suppose we're told that

$$\frac{\partial z}{\partial x} = \cos x \cos y, \qquad \frac{\partial z}{\partial y} = -\sin x \sin y, \qquad z = -1 \text{ when } x = 0 \text{ and } y = 0$$

Following the procedure from Example 1:

(1) $\frac{\partial z}{\partial x} = \cos x \cos y$ implies

$$z = \int \cos x \cos y \, dx + A(y) \overset{(*)}{=} \cos y \int \cos x \, dx + A(y) = \cos y \sin x + A(y)$$

for some function A of y. The equation $(*)$ holds because we treat y as constant in the integral.

(2) Differentiating the above with respect to y and comparing with $\frac{\partial z}{\partial y} = -\sin x \sin y$ gives

$$\frac{\partial z}{\partial y} = -\sin y \sin x + A'(y) = -\sin x \sin y$$

so $A'(y) = 0$ and $A(y) = C$ for some constant C.

(3) Thus $z = \sin x \cos y + C$ for some constant C. Setting $x = y = 0$ and using the initial values gives

$$z = -1 = \sin 0 \cos 0 + C = C$$

so $C = -1$.

In summary,

$$z = \sin x \cos y - 1$$