21-256: Matrices

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This is a summary of the important results about matrices that you should know.

Operations on matrices

- **Matrix addition.** If A and B are both $m \times n$ matrices then $A + B$ is the $m \times n$ matrix defined by
$$(A+B)_{ij} = A_{ij} + B_{ij} \quad \text{for } 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n$$
- **Scalar multiplication.** If A is an $m \times n$ matrix and λ is a scalar then λA is the $m \times n$ matrix defined by
$$(\lambda A)_{ij} = \lambda A_{ij} \quad \text{for } 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n$$
- **Transpose.** If A is an $m \times n$ matrix then A^T (the *transpose* of A) is the $n \times m$ matrix defined by
$$(A^T)_{ij} = A_{ji} \quad \text{for } 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$$
- **Matrix multiplication.** If A is an $m \times n$ matrix and B is an $n \times p$ matrix then AB is the $m \times p$ matrix defined by
$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj} \quad \text{for } 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant p$$
- **Inverse.** See below.

A matrix is *square* if it is an $n \times n$ matrix for some A. A square matrix A is *symmetric* if $A = A^T$.

The $n \times n$ *identity matrix* $I^{(n)}$ is defined by
$$(I^{(n)})_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

For example, $I^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The superscript $^{(n)}$ is dropped if it's clear from context.

If A is an $m \times n$ matrix then $I^{(m)}A = A = AI^{(n)}$.

Determinants

If A is an $n \times n$ matrix then $\widehat{A}_{ij}$ is the matrix with the i^{th} row and j^{th} colum deleted. For example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \Rightarrow \quad \widehat{A}_{23} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}$$

The determinant of a matrix is defined recursively:

- If A is a 1×1 matrix, say $A = \begin{pmatrix} a \end{pmatrix}$, then $\det(A) = a$.
- If A is an $n \times n$ matrix and $n > 1$ then, independently of the choice of row i, we have

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\widehat{A}_{ij})$$

In particular:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\begin{aligned} \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} &= a_1 \det \begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} - a_2 \det \begin{pmatrix} b_1 & b_3 \\ c_1 & c_3 \end{pmatrix} + a_3 \det \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \\ &= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1 \end{aligned}$$

We can expand along any row we like; the pattern of $+$ and $-$ signs follows:

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad \begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix} \qquad \cdots$$

Properties of the determinant include:

- $\det(AB) = (\det A)(\det B)$, so long as A and B are both square matrices of the same size;
- $\det(I) = 1$, and therefore $\det(A^{-1}) = \frac{1}{\det(A)}$;
- If A is an $n \times n$ matrix and λ is a scalar then $\det(\lambda A) = \lambda^n \det(A)$;
- $\det(A^T) = \det(A)$, meaning we can compute determinants by expanding down columns as well as along rows.

Inverses

If A is a square matrix, the *inverse* of A, if it exists, is the $n \times n$ matrix A^{-1} such that

$$AA^{-1} = I = A^{-1}A$$

First we need some definitions:

- If A is an $n \times n$ matrix then the $(i,j)^{\text{th}}$ *minor* of A is $M(A)_{ij} = \det(\widehat{A}_{ij})$.
- If A is an $n \times n$ matrix then the $(i,j)^{\text{th}}$ *cofactor* of A is $C(A) = (-1)^{i+j}M(A)_{ij}$.
- The cofactors of a matrix A define the *matrix of cofactors* $C(A)$.

Suppose A is a matrix. Then A^{-1} exists if and only if $\det(A) \neq 0$; and in this case

$$A^{-1} = \frac{1}{\det A} C(A)^T$$

Explicitly, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Determinants and vectors

We can extract information about vectors by writing the vectors as the column of a matrix. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are all m-dimensional vectors, then

$$(\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n)$$

is the $m \times n$ matrix whose i^{th} column is $\mathbf{v}_i$.

Theorem. If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are 3-dimensional vectors then

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \det(\mathbf{u} \mid \mathbf{v} \mid \mathbf{w})$$

Theorem. If $\mathbf{v}$ and $\mathbf{w}$ are 3-dimensional vectors then

$$\mathbf{v} \times \mathbf{w} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

Theorem. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are n-dimensional vectors then

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \text{ are LI} \quad \Leftrightarrow \quad \det(\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n) \neq 0$$

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \text{ are LD} \quad \Leftrightarrow \quad \det(\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n) = 0$$