21-256 Exam 3: Lagrange multipliers and integration (sample)

Friday 27th June, 10:30–11:20am

Instructions (please read carefully)

This test is split into two sections of five questions each:

- **Section A:** Lagrange multipliers
- **Section B:** Integration

You should attempt **exactly eight** questions, including **at least three** questions from each section. All questions will be marked out of 12. The duration of the test is 50 minutes. Please write legibly in the blue book provided. Calculators and other electronic devices are not permitted.

Please indicate on the table below which questions you attempted.

Question	Attempted? (✓)	Score
A1		
A2		
A3		
A4		
A5		
B1		
B2		
B3		
B4		
B5		
Free credit		4
Total		

Name: ____

Section A

For the following maximization (resp. minimization) problems, you may take for granted that if the vector equation $\boldsymbol{\nabla}\Lambda = \mathbf{0}$ has finitely many solutions, then the solution $(\bar{a}; \bar{\lambda})$ for which $f(\bar{a})$ is greatest (resp. least) is optimal.

A1. Find the values of (x, y, z) which maximize $x^4+y^4+z^4$ subject to the constraint $x^2+y^2+z^2 = 1$.

A2. Find the values of (x, y, z) which minimize $yz + xy$ subject to the constraints $x - y = 1$ and $y^2 - z^2 = 1$.

A3. Use the method of Lagrange multipliers to show that the rectangle of maximum area that has given perimeter p is a square.

A4. The plane $x + y + 2z = 2$ intersects the surface $z = x^2 + y^2$ in an ellipse. Find the point on this ellipse closest to the origin.

A5. A student wishes to maximize $f(x, y, z)$ subject to the constraints $g(x, y, z) = k$ and $h(x, y, z) = \ell$. She finds $(3, -3, 2; 0, 1)$ is a solution to $\boldsymbol{\nabla}\Lambda = \mathbf{0}$, and that the corresponding bordered Hessian is:

$$H = \begin{pmatrix} 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & -1 & 0 & -1 \\ 1 & -1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 \\ -3 & -1 & 0 & 0 & 2 \end{pmatrix}$$

(a) Why does $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ appear in the top left-hand corner of H?

(b) What conditions on determinants of sub-matrices of H must be satisfied in order to conclude from this that $f(3, -3, 2)$ is maximal?

Section B on next page

Section B

B1. Find the average value of y^2 on the disc of radius 2 centered at the origin in the xy-plane. You may use without proof the fact that

$$\frac{d}{dx}\left(6\sin^{-1}\frac{x}{2} - \frac{1}{4}x\sqrt{4-x^2}(x^2-10)\right) = (4-x^2)^{\frac{3}{2}}$$

B2. Find $\int_U 2x + 3y^2\,dA$, where U is the region of $\mathbb{R}^2$ defined by $y = x^2$ and $y = 8 - x^2$.

B3. Find the volume of the solid enclosed by the xy-plane, the yz-plane, the zx-plane and the plane $x + y + z = 4$.

B4. X and Y are independent continuous random variables whose probability density functions are defined by

$$f_X(x) = \begin{cases} k & \text{if } -1 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} \ell y^2 & \text{if } -2 \leqslant y \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the values of the constants k and ℓ, and find $\mathbb{P}(X \geqslant 0 \text{ and } |Y| \leqslant 1)$.

B5. A dartboard has radius 10 inches, and its bullseye has radius $\frac{1}{2}$ inch. I throw a dart at the board. The dart is twice as likely to land in the upper half of the board than the lower half, and twice as likely to hit the board than to miss it, but the dart never lands more than 1 inch from the board. Thus, I model x-coordinate X and y-coordinate Y of where it lands using the probability density function

$$g(x,y) = \begin{cases} 4c & \text{if } y \geqslant 0 \text{ and } x^2 + y^2 \leqslant 100 \\ 2c & \text{if } y < 0 \text{ and } x^2 + y^2 \leqslant 100 \\ 3c & \text{if } 100 < x^2 + y^2 \leqslant 121 \\ 0 & \text{otherwise} \end{cases}$$

What the value of the constant c? What is the probability that I hit the bullseye?

Please submit this question sheet with your answer booklet.